Some explicit aspects of modular forms over imaginary quadratic fields

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by

Adam Mohamed
from Ouani, Anjouan, Comoros

at

Universität Duisburg-Essen,
Faculty of Mathematics, Campus Essen

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Doctoral Committee:

Prof. Dr. Gabor Wiese
Prof. Dr. John Cremona.
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Declaration

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Contents

Introduction

In the later seventies F. Grunewald et. al, and J. Cremona have initiated the study of an explicit Langlands correspondence in the setting of imaginary quadratic fields. This pioneering work was done for the five imaginary quadratic fields equipped with an euclidean algorithm. Later their results were extended to imaginary quadratic fields without an euclidean algorithm and of higher class number. The present thesis is a modest contribution to that trend.

By explicit Langlands correspondences, we mean the explicit computation of modular forms, searching for modular elliptic curves over imaginary quadratic fields, explicit statement about mod p modular forms and their computation, explicit computation of Galois representations when applicable, testing when mod p Galois representations are modular, etc etc. Over the rational numbers many of the listed tasks are achieved in a very satisfactory manner by the standard computer algebra systems. In order to set up the context of the explicit Langlands correspondence, let us recall two of the great achievements in Arithmetic of the last two decades.

The celebrated Fermat's last theorem, or if you allow me, the Wiles-Taylor's theorem and its generalization by various arithmeticians, states that every rational elliptic curves is modular. Let $f(z) = \sum_{n>1}^{\infty} a_n(f)e^{2i\pi z}$ be a Hecke eigenform of weight two and level $N > 1$. Consider its associated L-function given as the Euler product

$$L(f,s) = \prod_{p, p\nmid N} (1 - a_p(f)p^{-s} + \mathbf{1}_N(p)p^{1-2s})^{-1}$$

where $\mathbf{1}_N(p) = 1$ if the rational prime p does not divide N and it is zero otherwise. Now let E be an elliptic over $\mathbb{Q}$ of conductor N. For $p \nmid N$, let $a_p(E)$ be the integer that counts the number of rational points of E viewed as a curve over $\mathbb{F}_p$:

$$a_p(E) = p + 1 - \sharp E(\mathbb{F}_p).$$

Then, a version of modularity says that there is a Hecke eigenform as above such that

$$a_p(f) = a_p(E).$$

In other words the Hasse-Weil L-function $L(E, s)$ associated to E is equal to $L(f, s)$.

The second proof of Fermat's last theorem is Serre's conjecture which is actually a theorem by Khare and Wintenberger. Let l be a prime number, then Deligne's theorem tells that for $f = \sum_{n>1}^{\infty} a_n(f) e^{2i\pi z}$ a newform of weight k and level $\Gamma_1(N)$, there exists an irreducible l-adic Galois representation

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_l)$$

which is unramified away from Nl and such that for each prime $p \nmid Nl$ the characteristic polynomial of $\rho(Frob_p)$ is

$$X^2 - a_p(f)X + \epsilon_f(p)p^{k-1}.$$

This is the l-adic Galois representation associated to f and it shall be denoted as ρ_f. Conversely let be given an irreducible, odd, mod l Galois representation

$$\bar{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_l).$$

Then Serre's conjecture asserts that there is a newform f of some weight and level such that $\bar{\rho} \cong \bar{\rho}_f$. As already said this is now a theorem. This illustrates another manifestation of modularity. All the objects appearing above are amenable for concrete computation. We also note the importance of the Fourier coefficients of Hecke eigenforms since they encode non-trivial information about the mod p rational points of rational elliptic curves. Naturally one asks for similar considerations when we take more general number fields. When the number field is totally real, this is the theory of Hilbert modular forms and this does not concern us here. What we are interested in is when the number field is a quadratic imaginary field. So, here is an overview of the main results of the present thesis.

Over imaginary quadratic fields it is more manegeable to view a modular form as a cohomology class. In particular a mod p cohomological modular form over an imaginary quadratic field F is a cohomology class with coefficients in some finite dimensional $\overline{\mathbb{F}}_p$-module. To be more precise let h denote the class number of F and $\mathcal{O}$ its ring of integers. The rational prime p is assumed to be inert in F and we fix an integral ideal $\mathfrak{n}$ which is coprime with p. Assume also that the positive generator of $\mathfrak{n} \cap \mathbb{Z}$ is greater than

3. Introduce the following open compact subgroup of level $\mathfrak{n}$

$$K_1(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \prod_{\mathfrak{q}\neq\infty} \mathrm{GL}_2(\mathcal{O}_{\mathfrak{q}}) : c, d-1 \in \mathfrak{n}\hat{\mathcal{O}} \right\}.$$

For the class group Cl of F, we choose representatives as follows. By the Chebotarev density theorem, we can choose representatives $[\mathfrak{b}_1], \cdots, [\mathfrak{b}_h]$ such that $[\mathfrak{b}_1] = [\mathcal{O}]$ and for $i > 1$, $[\mathfrak{b}_i]$ are prime ideals coprime with $\mathfrak{p}\mathfrak{n}$. So we have that $\mathfrak{b}_1$ corresponds to the idele $t_1 = 1$ and for $i > 1$, $\mathfrak{b}_i$ corresponds to the idele t_i with 1 in all places except at the $\mathfrak{b}_i$-place where we have a uniformizer of $\mathcal{O}_{\mathfrak{b}_i}$. Define $g_i = \left(\begin{smallmatrix} t_i & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Consider the following congruence subgroup of $\mathrm{GL}_2(F)$:

$$\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = g_i K_1(\mathfrak{n}) g_i^{-1} \cap \mathrm{GL}_2(F).$$

The Frobenius automorphism in $\mathrm{Gal}(\mathbb{F}_{p^2}/\mathbb{F}_p)$ is denoted by τ. We fix an embedding $\mathbb{F}_{p^2} \hookrightarrow \overline{\mathbb{F}}_p$. We consider the representation $V_{r,s}^{a,b}(\overline{\mathbb{F}}_p) = V_{r,s}^{a,b}(\mathcal{O}) \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p$, where $V_{r,s}^{a,b}(\mathcal{O}) = \mathrm{Sym}^r(\mathcal{O}^2) \otimes det^a \otimes (\mathrm{Sym}^s(\mathcal{O}^2))^\tau \otimes (det^b)^\tau$. So

$$V_{r,s}^{a,b}(\overline{\mathbb{F}}_p) = \mathrm{Sym}^r(\overline{\mathbb{F}}_p^2) \otimes det^a \otimes (\mathrm{Sym}^s(\overline{\mathbb{F}}_p^2))^\tau \otimes (det^b)^\tau.$$

We call classes in $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p))$ mod p cohomological modular forms. There are Hecke operators acting on $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p))$ and the Hecke eigenvalue systems are the mod p cohomological Hecke eigenforms. In Chapter 2 we prove the following theorem

Theorem 1. *Let F be an imaginary quadratic field of class number h. Let $\mathfrak{n}$ be an integral ideal in F and let $p > 5$ be a rational prime which is inert in F and coprime with $\mathfrak{n}$. Suppose that the positive generator of $\mathfrak{n} \cap \mathbb{Z}$ is greater than 3. Let $0 \leq r, s \leq p-1$ and $0 \leq l, t \leq p-1$, with l, t not both equal to $p-1$. Let ψ be a system of Hecke eigenvalues in $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p))$. Then ψ occurs in $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p \otimes det^{l+pt})$ except possibly when $(r = 1, s = p-2)$ or $(r = p-2, s = 1)$. In these potential exceptions, the system of eigenvalues is Eisenstein.*

This theorem seems to be known at least empirically since the existing experimental results about Serre's conjecture in our setting were obtained under the assumption that the conclusion in the theorem holds. Theorem 1 has the following consequence for Serre's conjecture over imaginary quadratic fields. Let $G_F := \mathrm{Gal}(\overline{F}/F)$ and let be given

$$\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

an irreducible mod p Galois representation of conductor $\mathfrak{n}$. Then the following Serre type question arises:

Conjecture 1 (Serre's conjecture)**.** *Does there exist a mod p cohomological Hecke eigenform of weight V and level $\mathfrak{n}$ with eigenvalues $\psi(T_\lambda)$ such that the trace $Tr(\rho(Frob_\lambda)) = \psi(T_\lambda)$ for all unramified prime ideals $\lambda \nmid \mathfrak{pn}$?*

In Chapter 2 we prove that

Proposition 1. *Let f be a mod p cohomological Hecke eigenform of weight V and level $\mathfrak{n}$. Then ρ is associated with f if and only if it is associated with some mod p cohomological Hecke eigenform of weight two ($\overline{\mathbb{F}}_p \otimes det^e$) for some $e \geq 0$ and level $\mathfrak{pn}$.*

The second part of the thesis is devoted to the study of Hecke operators on Manin symbols. Namely, Chapter 3 is concerned with the description of Hecke operators on Manin symbols. This is an improvement of the existing methods for computing eigenvalues of Hecke operators. Here we work with an imaginary quadratic field of class number one. So F has class number one and as above $\mathcal{O}$ is its ring of integers. We denote by $G := \mathrm{SL}_2(\mathcal{O})$. Let Γ be a congruence subgroup of G. Let R be an $\mathcal{O}$-module and consider V to be an $R[G]$-left module. Let $\mathcal{M}_R(\Gamma, V)$ be the space of modular symbols of weight V and level Γ, see in Chapter 3 for the precise definition. Denote by $M_R(\Gamma, V)$ the space of Manin symbols for Γ and of weight V. We fix a set of generators of G as follows. We define $T_1, \cdots, T_l$ as generators of G_∞, where G_∞ is the stabilizer subgroup for the linear fractional action of G on $\mathbb{P}^1(F)$. Next we complete the set $\{T_1, \cdots, T_l\}$ with matrices $\sigma_1, \cdots, \sigma_r \in G$ such that $G =< \sigma_1, \cdots, \sigma_r, T_1, \cdots, T_l : Relations >$ where "Relations" stands for the relations among the σ_i and T_j. We establish the following theorem

Theorem 2. *For $0 \leq i, j \leq r$ and $\theta \in Mat_2(\mathcal{O})_{\neq 0}$, let $a_{i,j,\theta} \in R$ satisfy Merel's C_Δ condition. Then the Hecke operator T_Δ on the Manin symbol $(0, \cdots, 0, g \otimes P, 0, \cdots, 0) \in \oplus_{i=1}^r R[\Gamma\backslash G] \otimes V$ with $g \otimes P$ in the i-th entry has j-th entry given as*

$$(T_\Delta.(0, \cdots, 0, g \otimes P, 0, \cdots, 0))_j = \sum_{\{\theta:\theta, g\theta \in \Delta^\iota G\}} a_{i,j,\theta}\psi(g\theta) \otimes \theta^\iota.P.$$

Theorem 2 is a generalization of a theorem of Merel to our setting. For the definitions of the Hecke operator T_Δ, Merel's C_Δ condition and the map ψ, we refer to Chapter 3. As one application of Theorem 2, we have the following description of Hecke operators on Manin symbols of weight V and level $\Gamma_1(\mathfrak{n})$

Proposition 2. *For all $1 \leq i,j \leq r$ and $\theta \in Mat_2(\mathcal{O})_\eta$, let $a_{i,j,\theta} \in R$ satisfy condition C_η. Then the Hecke operator T_η on $(0,\cdots,0,(u,v)\otimes P,0,\cdots,0)$ with $(u,v)\otimes P$ in the i-entry has j-entry given by*

$$(T_\eta.(0,\cdots,0,(u,v)\otimes P,0,\cdots,0))_j = \sum_{\theta\in g^{-1}\Delta_\eta^\iota G} a_{i,j,\theta}(u,v)\theta\otimes\theta^\iota.P.$$

For the description of Hecke operators on Manin symbols of weight V and level $\Gamma_0(\mathfrak{n})$, we have followed Cremona's simplification of Merel's description.

Proposition 3. *Consider the Manin symbol of level $\Gamma_0(\mathfrak{n})$ and weight V :*

$$(0,\cdots,0,\overbrace{(c:d)\otimes P}^{i\text{-th entry}},0,\cdots,0)\in\oplus_{i=1}^r R[\mathbb{P}^1(\mathcal{O}/\mathfrak{n})]\otimes V.$$

Let $a,b\in\mathcal{O}$ such that $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in G$ and $M=\begin{pmatrix} \delta & \beta\\ 0 & \eta/\delta\end{pmatrix}\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\begin{pmatrix} \eta/\delta' & -\beta'\\ 0 & \delta'\end{pmatrix}^{-1}\in G$. Let $\begin{pmatrix} \eta/\delta' & -\beta'\\ 0 & \delta'\end{pmatrix}(\{\sigma_i.\infty,\infty\}\otimes P)=\sum_{j=1}^r\sum_{k=1}^{s_j}M_{i,j,k}(\{\sigma_j.\infty,\infty\}\otimes M_{i,j,k}^{-1}\begin{pmatrix} \eta/\delta' & -\beta'\\ 0 & \delta'\end{pmatrix}.P)$. Then the j-th entry of the action of the matrix $\alpha=\begin{pmatrix} \delta & \beta\\ 0 & \eta/\delta\end{pmatrix}$ in the set R_η defining the Hecke operator T_η on $(0,\cdots,0,(c:d)\otimes P,0,\cdots,0)$ is given as follows

$$(\alpha.(0,\cdots,0,\overbrace{(c:d)\otimes P}^{i\text{-th entry}},0,\cdots,0))_j=\sum_{k=1}^{s_j}(c:d)\begin{pmatrix} \delta' & -\beta'\\ 0 & \eta/\delta'\end{pmatrix}M_{i,j,k}\otimes(\begin{pmatrix} \delta' & -\beta'\\ 0 & \eta/\delta'\end{pmatrix}M_{i,j,k})^\iota.P.$$

For the definition of the notation in Proposition 3, we refer to Chapter 3.

In the last Chapter 4 we construct families of matrices satisfying Merel's condition C_Δ and as one application of the theory of Hecke operators on Manin symbols described in Chapter 3, we obtain an interesting statement about L-functions associated to Hecke eigenforms. This is not the only direct consequence one can draw from the explicit description of Hecke operators on Manin symbols, but more importantly this gives us more freedom as we know now how to define explicit Hecke action on Manin symbols in compatibility with the Hecke action on modular symbols and hence on modular forms. As in the classical setting, the explicit description of Hecke operators on Manin symbols makes the computation of modular forms more efficient.

We are taking $\mathbb{C}$-coefficients. We consider the congruence subgroup of level one $G=\mathrm{SL}_2(\mathcal{O})$. The imaginary quadratic field F we are dealing with is one of the five euclidean imaginary quadratic fields. We can view the space of cuspidal modular forms over F of level G and weight $V_{r,s}(\mathbb{C})$ as the cohomology group $\mathrm{H}^1_{par}(G,V_{r,s}(\mathbb{C}))$. We denote the latter by H. The Hecke operators T_η defined by the sets χ_η of Heilbronn-Merel matrices also act on the space H. Let $\chi = \cup_{\eta \in \mathcal{O}} \chi_\eta$ where the union is taken up to units. Let $u \in \mathcal{O}^*$ be a unit. Let $J = \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$. There is a J-eigenspace with eigenvalue 1. We denote it as H^+. In Chapter 4 we prove the following.

Proposition 4. *Let $f \in H^+$ be an eigenform for all the Hecke operators $T_{\mathfrak{a}}$. Let $\mathfrak{L} \in (H^+)^\vee$, the $\mathbb{C}$-dual of H^+ and suppose that $\mathfrak{L}(f) \neq 0$. Then the formal L-series associated with $\mathfrak{L}$ and f defined by*

$$L_{\mathfrak{L},f}(s) = \sum_{M \in \chi} \frac{\mathfrak{L}(M.f)}{N(det(M))^s}$$

is up to a factor the L-series associated with the cuspidal eigenform f for the congruence subgroup G. The constant factor being $\mathfrak{L}(f)$.

Because the statement in Proposition 4 is the equivalent of Merel's universal Fourier expansion of modular forms, Proposition 4 is referred to as a universal Hecke L-series associated with cuspidal eigenforms.

In Chapter 4 we provide some data computed by Mehmet Haluk Şengün which demonstrates that our description of Hecke operators on Manin symbols greatly improves computation of Hecke eigenvalue systems.

The short Chapter 1 is a brief overview of modular forms over imaginary quadratic fields. Chapter 2 is independent from the other chapters. The notation introduced in Chapter 2 is independent from the notation introduced in the chapters onward. We have tried our best to avoid clashes between notation of the different chapters.

Chapter 1

Modular forms over imaginary quadratic fields

This is a very brief summary of some of the facts we need to know about modular forms over imaginary quadratic fields. These are all gathered from various resources and mostly facts are stated without proofs.

1.1 Introduction

The general theory of modular forms or automorphic forms for GL_2 over number fields is a well established theory, see for instance [38]. In particular, when the number field is totally real, then we are dealing with Hilbert modular forms and there is a vast amount of work concerning their theoretical and computational aspects. In here many of the fundamental arithmetical results available when the number field is the rational field $\mathbb{Q}$ ought to be extended to totally real fields among other things. This occupies a lot of mathematicians nowdays.

When the number field is an imaginary quadratic field, then peculiar difficulties arise. The main reason for this being that unlike in the totally real case, the symmetric space that one has to consider is a 3 dimensional real space and so one has no complex structure in hand. Despite this fact many of the fundamental arithmetical results known over $\mathbb{Q}$ are believed to have their counterpart over imaginary quadratic fields. This is why there is an interest in developing computational methods for automorphic forms over imaginary quadratic fields, see for instance [19], or [15], or [42].

There are at least three ways one can think of modular form over imaginary quadratic fields. Amid these, there are two dry ways namely, one can view them as automorphic forms on GL_2 of the adeles, or they can be considered as real analytic functions on the 3-dimensional hyperbolic space $\mathbb{H}_3$. Lastly they can be seen as cohomological classes of a congruence subgroup. As such they seem more manageable for our theoretical and computational investigations.

Here we very briefly review hyperbolic 3-space, automorphic forms on the adeles and corresponding modular forms on the hyperbolic space.

1.2 Hyperbolic 3-space $\mathbb{H}_3$

We shall define three dimensional hyperbolic space $\mathbb{H}_3$, the symmetric space associated with an imaginary quadratic field F.

Hyperbolic 3-space

Let F be an imaginary quadratic field of class number h and $\mathcal{O}$ its ring of integers. Let $\mathrm{GL}_2(\mathbb{C})$ and $\mathrm{SL}_2(\mathbb{C})$ be the general linear and the special linear group with entries in the field of complex numbers. For most of what we will say in this subsection we refer to [18] and the references therein for more details.

Definition 1.2.1. *The hyperbolic 3-space which we denoted as $\mathbb{H}_3$ is the space*

$$\mathbb{H}_3 := \mathbb{C} \times \mathbb{R}_{>0} = \{(z, r) : z \in \mathbb{C}, r \in \mathbb{R}, r > 0\}.$$

Actually this is the model of 3-dimensional hyperbolic space in Euclidean three-space. As such it is the equivalent of the classical upper half plane model $\mathbb{H}_2 := \{x + yi \in \mathbb{C} : y > 0\}$ of 2-dimensional hyperbolic space. Naturally $\mathbb{H}_2$ sits inside $\mathbb{H}_3$. For more conveniences in the formulas, it is better to view $\mathbb{H}_3$ as a subspace of the skew field of quaternions $\mathbb{H}$ with basis over $\mathbb{R}$ being given as $1, i, j, k$. The injection from $\mathbb{H}_3$ into $\mathbb{H}$ is given as

$$\begin{array}{ccc} \mathbb{H}_3 & \to & \mathbb{H} \\ (z, r) & \mapsto & z + rj. \end{array}$$

The group $\mathrm{GL}_2(\mathbb{C})$ acts on $\mathbb{H}_3$ as follows. Viewing $P \in \mathbb{C}$ as a point in $\mathbb{H}$ then a matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$ transforms P as

$$M.P = (aP + b)(cP + d)^{-1}.$$

Here the inverse is taken in the skew field of quaternions. More explicitly when we write $P = z + rj$, and denote $\bar{}$ as the complex conjugation, then $M.P = \tilde{z} + \tilde{r}j$, where

$$\begin{aligned} \tilde{z} &= \frac{(az+b)(\bar{c}\bar{z}+\bar{d}) + a\bar{c}r^2}{|cz+d|^2 + |c|^2r^2} \\ \tilde{r} &= \frac{|\,det(M)\,|r}{|cz+d|^2 + |c|^2r^2}. \end{aligned}$$

The stabilizer of the point $j = (0,0,1) \in \mathbb{H}_3$ for the action of $\mathrm{GL}_2(\mathbb{C})$ we just described is the special unitary subgroup of $\mathrm{GL}_2(\mathbb{C})$, $\mathrm{U}_2(\mathbb{C})$. Therefore the description of $\mathbb{H}_3$ as a symmetric space is given as

$$\begin{aligned} \mathrm{U}_2(\mathbb{C})\backslash\mathrm{GL}_2(\mathbb{C}) &\longleftrightarrow \mathbb{H}_3 \\ g &\mapsto g.j. \end{aligned}$$

Recall also the classical action of $\mathrm{GL}_2(\mathbb{C})$ on the Riemann sphere $\mathbb{P}^1(\mathbb{C})$. It is defined by linear fractional transformation: for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$ and $z \in \mathbb{P}^1(\mathbb{C})$ we have

$$g.z = \frac{az+b}{cz+d}.$$

For our purposes, the groups we shall be dealing with are arithmetic subgroups of $\mathrm{GL}_2(F)$. These groups are also known as *Bianchi groups*. These are the subgroups of $\mathrm{GL}_2(\mathbb{C})$ that act discontinuously on the hyperbolic three space $\mathbb{H}_3$. A group Γ acting on $\mathbb{H}_3$ is said to be acting discontinuously if and only if for any compact subset K of $\mathbb{H}_3$, the set

$$\{\gamma \in \Gamma : \gamma K \cap K \neq \emptyset\}$$

is finite. For F an imaginary quadratic field of class number h with $\mathcal{O}$ as integer ring, there is an action of $\mathrm{GL}_2(\mathcal{O})$ on the *set of cusps* $\mathbb{P}^1(F)$ by linear fractional transformation. Let us write a cusp as $(\alpha : \beta)$ and denote by $<\alpha, \beta> := \alpha\mathcal{O} + \beta\mathcal{O}$, the fractional ideal generated by α and β.

Proposition 1.2.2. *Let Cl be the class group of F. Then, there is a bijection*

$$\begin{aligned} \mathrm{GL}_2(\mathcal{O})\backslash\mathbb{P}^1(F) &\longleftrightarrow Cl \\ (\alpha:\beta) &\mapsto [<\alpha,\beta>]. \end{aligned}$$

Proof. The map is surjective because any fractional ideal of F can be generated by two elements. Now we have to see that two cusps generating the same fractional ideal class are congruent modulo $\mathrm{GL}_2(\mathcal{O})$. So let $\mathfrak{a} =< \alpha, \beta >$, $\mathfrak{b} =< \lambda, \gamma >$ and suppose that we have $[\mathfrak{a}] = [\mathfrak{b}]$. Because of the equality of ideal classes, there are c and d in F^* such that we have equality in term of ideals: $< c\alpha, c\beta >=< d\lambda, d\gamma >$. Therefore there is an invertible integral matrix transforming the vector $(d\lambda, d\gamma)$ to $(c\alpha, c\beta)$ and reciprocally. By definition of the space $\mathbb{P}^1(F)$, we deduce that the statement in the proposition is valid. □

For other notions such as fundamental domain for the action of $\mathrm{GL}_2(\mathcal{O})$, manifolds structure on $\mathbb{H}_3$ and some related concepts we refer to [18].

1.3 Adelic modular forms and modular forms on $\mathbb{H}_3$

For more details about the facts we shall mention in this subsection we refer to [10]. In there there is a wealthy amount of material about most of all the background one needs for the explicit theory of modular forms over imaginary quadratic fields. For more general considerations, see [38]. We shall emphasize that we will not use any of the facts contained in this subsection. Again, Bygott has given a very elaborate treatment of the subject.

1.3.1 Adelic modular forms

Let $\mathbb{A}$ be the ring of adeles of F and consider and $G := \mathrm{GL}_2$ the linear algebraic group. Let $\mathfrak{n}$ be a non-zero ideal of $\mathcal{O}$. Let $Z_{\mathbb{A}}$ be the center of $G(\mathbb{A})$. For a finite place $\mathfrak{q}$ of F, define the following congruence subgroup of $G(\mathcal{O}_{\mathfrak{q}})$:

$$K_{0,\mathfrak{q}}(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(\mathcal{O}_{\mathfrak{q}}) : c \in \mathfrak{n}\mathcal{O}_{\mathfrak{q}} \right\}.$$

Let $\Gamma_0(\mathfrak{n})$ be the congruence subgroup of $\mathrm{GL}_2(\mathcal{O})$ of the matrices that reduce modulo $\mathfrak{n}$ to elements from the Borel subgroup of upper triangular matrices with coefficients in $\mathcal{O}/\mathfrak{n}$. Let $\psi : F^*\backslash\mathbb{A}^* \to \mathbb{C}^*$ be a quasicharacter of the idele class group of conductor dividing $\mathfrak{n}$. The characters induced by ψ on $F_{\mathfrak{q}}^*$ and $\mathbb{C}^*$ are denoted as $\psi_{\mathfrak{q}}$ and ψ_∞ respectively.

Consider a finite dimensional $\mathbb{C}$-vector space V and let $\rho : \mathrm{U}_2 \to \mathrm{GL}(V)$ be an irreducible representation of the unitary subgroup of $G(\mathbb{C})$ which agrees with ψ on the center Z_∞ of U_2. This is referred to as a weight; see [10] for the reason why ρ is called a weight. We consider the space of functions $\Psi : G(\mathbb{A}) \to V$ satisfying the following conditions:

(a) $\Psi(\lambda g) = \Psi(g)$ for all $\lambda \in G(F)$ and $g \in G(\mathbb{A})$; that is Ψ is $G(F)$-left invariant

(b) $\Psi(g\zeta) = \Psi(g)\psi(\zeta)$ for all $g \in G(\mathbb{A})$ and $\zeta \in Z_{\mathbb{A}}$

(c) $\Psi(g\kappa) = \Psi(g)$ for all $g \in G(\mathbb{A})$ and $\kappa \in \prod' K_{0,\mathfrak{q}}(\mathfrak{n})$ (the restricted product being over all finite places $\mathfrak{q}$ away from $\mathfrak{n}$); that is Ψ is right invariant on an open compact subgroup of $G(\mathbb{A}_f)$

(d) for $\mathfrak{q} \mid \mathfrak{n}$ and for all $g \in G(\mathbb{A})$ and $\kappa = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_{0,\mathfrak{q}}(\mathfrak{n})$, $\Psi(g\kappa) = \Psi(g)\psi_{\mathfrak{q}}(d)$

(e) $\Psi(g\kappa) = \Psi(g)\rho(\kappa)$ for all $g \in G(\mathbb{A})$ and $\kappa \in \mathrm{U}_2$.

Functions satisfying conditions (a) to (d) have a Fourier expansion as follows.

1.3.2 Fourier expansion

One introduces the following subgroup of $G(\mathbb{A})$ which we denote as $\mathbb{U}$:

$$\mathbb{U} = \left\{ \left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \in G(\mathbb{A}) \right\}.$$

One also defines $K_0(\mathfrak{n}) = \prod_{\mathfrak{q} \nmid \infty} K_{0,\mathfrak{q}}(\mathfrak{n})$, an open compact subgroup of $G(\hat{\mathcal{O}})$ of level $\mathfrak{n}$. There is the following decomposition of $G(\mathbb{A})$, see [38] for details:

$$G(\mathbb{A}) = G(F)\mathbb{U}K_0(\mathfrak{n})Z_{\mathbb{A}}.$$

This decomposition implies that any function from $\Psi : G(\mathbb{A}) \to V$ which satisfies the conditions (a) to (e) is uniquely given by its restriction to $\mathbb{U}$. This allows one to define a function $\mathcal{F}$ on $\mathbb{A} \times \mathbb{A}^*$ by putting:

$$\mathcal{F}(x, y) := \Psi\big(\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right)\big).$$

Let now ϕ be an additive character of $\mathbb{A}$ which is trivial on F. We fix an idele $\delta \in \mathbb{A}^*$ corresponding to the different of F.

Theorem 1.3.1. *Let $\Psi : G(\mathbb{A}) \to V$ satisfying conditions (a) to (d). Then $\mathcal{F}$ has a Fourier expansion given as*

$$\mathcal{F}(x, y) = c_0(y) + \sum_{\xi \in F^*} c(\xi\delta y)\phi(\xi x)$$

where $c_0(\eta y) = c_0(y)$ for all $\eta \in F^*$, $c_0(uy) = c_0(y)$ for all $u \in \prod_{\mathfrak{q}} \mathcal{O}_{\mathfrak{q}}^*$, the Fourier coefficient $c(y)$ only depends on y_∞ and the ideal corresponding to y; lastly $c(y) = 0$ unless that ideal is integral.

Proof. This is Proposition 95 from [10, p. 144]. □

Definition 1.3.2 (Cuspidality condition)**.** *Let $\Psi : G(\mathbb{A}) \to V$ be given by a Fourier expansion as in Theorem 1.3.1. Then one says that Ψ is cuspidal if and only if for all $y \in \mathbb{A}^*$ we have $c_0(y) = 0$.*

There are further analytical conditions that come into play in order to have a working theory. The first one is a growth condition and the other one is the harmonicity condition. We shall state the growth condition and put under the carpet the harmonicity condition. For details on the latter condition we refer to [10] and [38]. Let $\| \ \|$ be a fixed norm on V.

Definition 1.3.3. *Let $\Psi : G(\mathbb{A}) \to V$ satisfy the conditions (a) to (e). We say that Ψ is $\mathbb{U}$-moderate if there exist constants $C > 0$ and $\lambda \geq 0$ such that for all $x \in \mathbb{A}, y \in \mathbb{A}^*$ we have*

$$\| \ \Psi\big(\left(\begin{smallmatrix} y & x \\ 0 & 1 \end{smallmatrix}\right) \big) \| \leq Csup(|\, y\,|^{\lambda}, |\, y\,|^{-\lambda}).$$

Definition 1.3.4 (Adelic automorphic form and cusp form)**.** *An automorphic form of weight ρ, character ψ for $\Gamma_0(\mathfrak{n})$, is a function $\Psi : G(\mathbb{A}) \to V$ that satisfies conditions $(a) - (e)$ and is $\mathbb{U}$-moderate and holomorphic. A cusp form is a cuspidal automorphic form.*

Now we shall say how from automorphic forms on $G(\mathbb{A})$ one gets modular forms on the homogeneous space $\mathbb{H}_3$.

1.3.3 From $G(\mathbb{A})$ to $\mathbb{H}_3$

Take representatives $[\mathfrak{b}_i], \cdots, [\mathfrak{b}_h]$ of the ideal class group of F such that $[\mathfrak{b}_1] = [\mathcal{O}]$ and for $i > 1$, the ideals $\mathfrak{b}_i$ are integral prime ideals coprime with $\mathfrak{n}$. To these representatives correspond finite ideles t_i with $t_1 = 1$, the idele with 1 in all places and $t_i = (1, \cdots, \overbrace{\pi_i}^{\mathfrak{b}_i\text{- th place}}, 1, \cdots, 1, \cdots)$ where π_i is a uniformizer of $\mathcal{O}_{\mathfrak{b}_i}$. Define $g_i = \left(\begin{smallmatrix} t_i & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Let $\Omega_0(\mathfrak{n}) = G(\mathbb{C}) \times K_0(\mathfrak{n})$. Define also the following congruence subgroups

$$\Gamma_{[\mathfrak{b}_i]}(\mathfrak{n}) := g_i \Omega_0(\mathfrak{n}) g_i^{-1} \cap G(F).$$

There is the following decomposition of $G(\mathbb{A})$:

$$G(\mathbb{A}) = \amalg G(F) g_i \Omega_0(\mathfrak{n}).$$

Let ψ be a quasicharacter as above with conductor dividing $\mathfrak{n}$. This quasicharacter ψ induces a Dirichlet character χ of $(\mathcal{O}/\mathfrak{n})^*$ given by

$$\chi(d) = \prod_{\mathfrak{q}|\mathfrak{n}} \psi_{\mathfrak{q}}(d).$$

This is Lemma 71 from [10, p. 117]. Next, one introduces a character of $K_0(\mathfrak{n})$ by setting

$$\begin{aligned} \hat{\psi} : K_0(\mathfrak{n}) &\rightarrow \mathbb{C}^* \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &\mapsto \prod_{\mathfrak{q}|\mathfrak{n}} \psi_{\mathfrak{q}}(d). \end{aligned}$$

Take $\Psi : G(\mathbb{A}) \rightarrow V$ satisfying the symmetric conditions (a) to (e). Since Ψ is $G(F)$-left invariant we have well defined h-tuples of functions on $\Omega_0(\mathfrak{n})$:

$$\begin{aligned} \Psi_i : \Omega_0(\mathfrak{n}) &\rightarrow V \\ \omega &\mapsto \Psi(g_i\omega). \end{aligned}$$

For $a \in \mathbb{A}$ we write a_∞ and a_f for the infinite part of a and the finite part of a respectively. Because of the $K_0(\mathfrak{n})$-right invariance of Ψ_i, one can define a function on $G(\mathbb{C})$ by putting

$$\begin{aligned} \phi_i : G(\mathbb{C}) &\rightarrow V \\ \delta &\mapsto \Psi_i((g_i^{-1}\delta g_i)\infty, 1). \end{aligned}$$

This function is $\Gamma_{[\mathfrak{b}_i]}$-left invariant. Conversely given a function $\phi_j : G(\mathbb{A}) \rightarrow V$ which is $\Gamma_{[\mathfrak{b}_j]}$-left invariant one defines a function Ψ_j on $\Omega_0(\mathfrak{n})$ by putting

$$\begin{aligned} \Psi_j : \Omega_0(\mathfrak{n}) &\rightarrow V \\ x &\mapsto \phi_j((g_j x g_j^{-1})_\infty)\hat{\psi}(x_f). \end{aligned}$$

Theorem 1.3.5. *There is a bijection between the set of functions Ψ satisfying conditions $(a), (c)$ and (d); and the set of h-tuples of functions ϕ_i on $G(\mathbb{C})$ such that for $1 \leq i \leq h$ and for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_{[\mathfrak{b}_i]}(\mathfrak{n})$ and $\delta \in G(\mathbb{C})$, one has*

$$\phi_i(\gamma\delta) = \phi_i(\delta).\chi^{-1}(d).$$

Proof. This is Theorem 98 from [10, p. 150]. $\square$

Given a function ϕ_i as in Theorem 1.3.5, on defines functions on $\mathbb{H}_3$ by setting

$$\begin{array}{rcl} f_i : \mathbb{H}_3 & \to & V \\ (z,t) & \mapsto & \phi_i\big(\left(\begin{smallmatrix} t & z \\ 0 & 1 \end{smallmatrix}\right) \big). \end{array}$$

So there is a one to one correspondence between automorphic forms on $G(\mathbb{A})$ and the set of h-tuples of functions on $\mathbb{H}_3$ satisfying some symmetric properties under the action of the congruence subgroups $\Gamma_{[\mathfrak{b}_i]}(\mathfrak{n})$. In order to give some of the available examples we need to take $V = \mathbb{C}^k$ for some $k > 0$. For an explanation of why we can consider this V, we refer to [10].

One example from [15]

For this example we assume that F has class number one and we shall speak about weight 2 cusp forms for the standard congruence subgroup $\Gamma_0(\mathfrak{n})$. In fact a cusp form of weight 2 is a vector valued function $\mathcal{F} : \mathbb{H}_3 \to \mathbb{C}^3; (z,r) \mapsto (F_0, F_1, F_3)$ such that the differential form $-\mathcal{F}_0 \frac{dz}{t} + \mathcal{F}_1 \frac{dt}{t} + \mathcal{F}_3 \frac{d\bar{z}}{t}$ is a $\Gamma_0(\mathfrak{n})$-invariant harmonic differential on $\mathbb{H}_3$ that is well behaved at the cusp. A function satisfying theses properties admits a Fourier expansion about the cusp $(0, \infty)$ of the form

$$\mathcal{F}(z,t) = \frac{16\pi^2 w_F}{D_F} \sum_{0 \neq \mathfrak{a} < \mathcal{O}} c(\mathfrak{a}) t \mathbf{K}(4\pi|\eta|^{-1}|\alpha|t)\psi(\eta^{-1}\alpha z)$$

where D_F denotes the discriminant of F, w_F the number of units of F, η generates the different of F, $\psi(z) = exp(2\pi i(z + \bar{z}))$, the normalized complex additive character, and $\mathbf{K}$ is the vector valued F-Bessel function

$$\mathbf{K}(t) = (-\frac{1}{2}K_1(t), K_0(t), \frac{1}{2}iK_1(t)).$$

To see examples of Eisenstein series, we refer to [19].

1.3.4 Correspondence with elliptic curves over F

Let $\mathfrak{n}$ be an integral ideal and consider $\Gamma_0(\mathfrak{n})$. Let f be a weight two eigenform of level $\Gamma_0(\mathfrak{n})$ for all the Hecke operators $T_\mathfrak{q}$ such that $T_\mathfrak{q} = c_\mathfrak{q}(f)f$ where $c(\mathfrak{q})$ is rational. The L-series associated with f is defined as

$$L(f,s) = \sum_{0 \neq \mathfrak{a} \lhd \mathcal{O}} c_\mathfrak{a}(f) N(\mathfrak{a})^{-s}$$

where s is complex number and N denote the norm map. This L-series converges in some half space and satisfies functional equation, Euler product, admits an analytic continuation to the whole complex plane.

Given an elliptic curve E of conductor $\mathfrak{n}$ defined over F, there is also a corresponding L-series of the form

$$L(E,s) = \sum_{0 \neq \mathfrak{a} \lhd \mathcal{O}} c_{\mathfrak{a}}(E) N(\mathfrak{a})^{-s}.$$

This L-series is known to converge when the real part $\Re(s)$ of s is strictly greater than $\frac{3}{2}$. But it is not known if $L(E,s)$ has an analytic continuation and admits a functional equation. The principal known approach to this king of questions is to relate $L(f,s)$ with $L(E,s)$. This is one aspect of modularity of elliptic curves. The Taylor-Wiles methods does not apply in this setting essentially because the symmetric space associated with $\mathbb{H}_3$ does not have a complex structure. Nonetheless, there is computational evidence that suggest that L-series of modular forms over F on one hand and L-series of elliptic curves over F are related. For such an example we refer to [15] and [35]. Therefore any kind of input towards the improvement of the current computational method (the modular symbols algorithm) is useful.

1.4 Eichler-Shimura Harder isomorphism

In this section we assume that the imaginary quadratic fields F is of class number one. We denote by σ the non-trivial element in the Galois group $\mathrm{Gal}(F/\mathbb{Q})$. We fix an embedding of F into $\mathbb{C}$. We consider the following $\mathbb{C}[\mathrm{GL}_2(\mathcal{O})]$-representations:

$$V_{r,s}(\mathbb{C}) := \mathrm{Sym}^r(\mathbb{C}^2) \otimes \mathrm{Sym}^s(\mathbb{C}^2)^{\tau} = \mathbb{C}[X,Y]_r \otimes \mathbb{C}[X,Y]_s^{\tau}.$$

Here $\mathbb{C}[X,Y]_r$ is the ring of homogeneous polynomials of degree r in the variables X, Y. Let Γ be a congruence subgroup of $\mathrm{SL}_2(F)$ which is torsion free. Let $\mathcal{V}_{r,s}(\mathbb{C})$ be the locally constant sheaf of $\mathbb{C}$-vector spaces on $\Gamma\backslash\mathbb{H}_3$ associated with $V_{r,s}(\mathbb{C})$. Let $U(\Gamma)$ be the closure of Γ in $\mathrm{SL}_2(\mathbb{A}_f)$, that is $U(\Gamma)$ is such that $\Gamma = U(\Gamma) \cap \mathrm{SL}_2(F)$. Let $S_n(\Gamma, \mathbb{C}) := \oplus \rho_f^{U(\Gamma)}$, where the sum is over all the cuspidal automorphic representations $\rho = \rho_f \otimes \rho_\infty$ of SL_2 over F with ρ_∞ the principal series representation of $\mathrm{SL}_2(\mathbb{C})$ induced by the character: $\left(\begin{smallmatrix} a & * \\ 0 & a^{-1} \end{smallmatrix}\right) \mapsto (\frac{a}{|a|})^{2n+2}$. The equivalent of Eichler-Shimura theorem in our context is as follows.

Theorem 1.4.1 (Harder). *1. For all r, s and $i > 2$, we have $\mathrm{H}^i_{cusp}(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,s}(\mathbb{C})) = 0$. For $1 \leq i \leq 2$, $\mathrm{H}^i_{cusp}(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,s}(\mathbb{C})) = 0$ unless $r = s$.*

2. $\mathrm{H}^1_{cusp}(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,r}(\mathbb{C})) \cong \mathrm{H}^2_{cusp}(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,r}(\mathbb{C})) \cong S_r(\Gamma, \mathbb{C})$.

3. $\mathrm{H}^0_{Eis}(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,s}(\mathbb{C})) = 0$ *unless* $r = s$ *in which case it is* $\mathbb{C}$.

Fortunately the space $\mathrm{H}^1(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,r}(\mathbb{C}))$ can be identified with the group cohomology $\mathrm{H}^1(\Gamma, V_{r,s}(\mathbb{C}))$. In this thesis, we shall be working with mod p cohomological modular forms, that is to mean we will consider the $\overline{\mathbb{F}}_p$-vector space $\mathrm{H}^1(\Gamma, V_{r,s}(\overline{\mathbb{F}}_p))$ or finite sums of such spaces. And more generally, we will also be working with the spaces $\mathrm{H}^1(\Gamma, V_{r,s}(A))$ or $\mathrm{H}^2(\Gamma, V_{r,s}(A))$ with A an $\mathcal{O}$-algebra.

Chapter 2

Weight reduction for mod p modular forms

Let F be an imaginary quadratic field and $\mathcal{O}$ its ring of integers . Let $\mathfrak{n} \subset \mathcal{O}$ be a non-zero ideal and let $p > 5$ be a rational inert prime in F and coprime with $\mathfrak{n}$. Let V be an irreducible finite dimensional representation of $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{F}_{p^2})]$. We establish that a system of Hecke eigenvalues appearing in the cohomology with coefficients in V already lives in the cohomology with coefficients in $\overline{\mathbb{F}}_p \otimes det^e$ for some $e \geq 0$; except possibly in some few cases.

2.1 Introduction

Let F be an imaginary quadratic field with $\mathcal{O}$ as its ring of integers. The class number of F is denoted as h. Let Γ be a congruence subgroup of $\mathrm{GL}_2(\mathcal{O})$. Let σ be the non-trivial element of $\mathrm{Gal}(F/\mathbb{Q})$. We consider the representations of $\mathrm{GL}_2(\mathcal{O})$ defined as $V_{r,s}^{a,b}(\mathcal{O}) = \mathrm{Sym}^r(\mathcal{O}^2) \otimes det^a \otimes (\mathrm{Sym}^s(\mathcal{O}^2))^\sigma \otimes (det^b)^\sigma$ where a, b, r, s are positive integers. For an $\mathcal{O}$-algebra A, we define $V_{r,s}^{a,b}(A) := V_{r,s}^{a,b}(\mathcal{O}) \otimes_{\mathcal{O}} A$. A cohomological modular form of level Γ and weight $V_{r,s}^{a,b}(A)$ over F is a class in $\mathrm{H}^1(\Gamma, V_{r,s}^{a,b}(A))$. As in the classical setting, the space $\mathrm{H}^1(\Gamma, V_{r,s}^{a,b}(A))$ can be endowed with a structure of Hecke module. The Hecke algebra acting on $\mathrm{H}^1(\Gamma, V_{r,s}^{a,b}(A))$ is commutative and has its elements indexed over the integral ideals of F. So, one can consider eigenclasses or eigenforms which are eigenvectors for all the Hecke operators $T_{\mathfrak{a}}$. Hence to such an eigenform corresponds a system of Hecke eigenvalues.

Integral systems of eigenvalues when reduced modulo a prime p are believed to be related to mod p representations of Galois groups as conjectured by Ash et al. in [2]. One instance of this correspondence being the theorem of Deligne constructing l-adic representations of the absolute Galois group of $\mathbb{Q}$, $G_{\mathbb{Q}} := Gal(\overline{\mathbb{Q}}/\mathbb{Q})$, via systems of Hecke eigenvalues arising from modular forms over $\mathbb{Q}$. Let N be a positive integer and $\Gamma_0(N)$ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. Take V to be the $\mathrm{SL}_2(\mathbb{Z})$-module given as $V := Sym^{k-2}(\mathbb{Z}^2) = \mathbb{Z}[X,Y]_{k-2}$, the space of homogeneous polynomials of degree $k-2$ over $\mathbb{Z}$ in two variables and with k even. The converse of Deligne's theorem, Serre's modularity conjecture, which is now a theorem of Khare and Wintenberger, has been formulated in the language of group cohomology in [3] and the standard conjecture in there relates mod p Galois representations of $G_{\mathbb{Q}}$ to systems of Hecke eigenvalues on $\mathrm{H}^1(\Gamma_0(N), V \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p)$.

Next let N and n be positive integers. In [4], it was shown that a system of Hecke eigenvalues occurring in the cohomology of $\Gamma_1(N)$ with coefficients in some $\mathrm{GL}_n(\mathbb{F}_p)$-module also occurs in the cohomology with coefficients in some irreducible $\mathrm{GL}_n(\mathbb{F}_p)$-module. This fact has some interesting features. In fact it allows one to obtain a cohomological avatar of the so-called Hasse invariant, see [17]. That is, one can produce congruences between weight two and higher weight modular forms using cohomological methods.

As for the case of an imaginary quadratic field F of class number one, then when p splits in F and is coprime with $\mathfrak{n}$, in [29], it is established that a Hecke system of eigenvalues occurring in the first cohomology with non-trivial coefficients can be realized in the first cohomology with trivial coefficients. This should also hold when the class number of F is greater than one.

Let p be a rational prime coprime to $\mathfrak{n}$ and inert in F. Let E be a finite dimensional representation of $\mathrm{GL}_2(\mathbb{F}_{\mathrm{p}^2})$ over $\overline{\mathbb{F}}_p$. Let Γ be a congruence subgroup of $\mathrm{GL}_2(\mathcal{O})$. Then a cohomological mod p modular form of level Γ and weight E is defined to be a class in $\mathrm{H}^1(\Gamma, \mathrm{E})$. As in the classical setting there is a Hecke algebra action on the space $\mathrm{H}^1(\Gamma, \mathrm{E})$ and one can consider systems of Hecke eigenvalues for the space $\mathrm{H}^1(\Gamma, \mathrm{E})$. Our aim will be to say something more precise about systems of Hecke eigenvalues in this setting. We will prove that a system of Hecke eigenvalues living in $\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), M)$ where M is an irreducible $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{F}_{\mathrm{p}^2})]$-module also occurs in $\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p \otimes det^e)$ for some $e \geq 0$ depending on M; except possibly for some cases. See Theorem 2.4.11 for the precise statement. Here $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ are some congruence subgroups defined in Section 2.3.

There is an application of Theorem 2.4.11 related to Serre type questions about mod p Galois representations of the absolute Galois group of F. When we are dealing with cohomological modular forms mod p with trivial coefficients $\overline{\mathbb{F}}_p$, we shall say that we are in *weight two*. Let $G_F := \mathrm{Gal}(\overline{F}/F)$ and let be given

$$\rho : G_F \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

an irreducible mod p Galois representation of conductor $\mathfrak{n}$. Let Tr denotes the trace of a matrix. Then the following questions arise:

(a) Does there exist a cohomological Hecke eigenform of some weight V and level $\mathfrak{n}$ with eigenvalues $\psi(T_\lambda)$ such that $Tr(\rho(Frob_\lambda)) = \psi(T_\lambda)$ for all unramified prime ideals $\lambda \nmid \mathfrak{pn}$?

(b) Does there exist a cohomological Hecke eigenform of weight 2 ($V = \overline{\mathbb{F}}_p \otimes det^e$ for some $e \geq 0$) and level $\mathfrak{pn}$ with $Tr(\rho(Frob_\lambda)) = \psi(T_\lambda)$ for all unramified prime ideals $\lambda \nmid \mathfrak{pn}$?

As a consequence of Theorem 2.4.11, we shall see that the two questions above are equivalent. See Proposition 2.4.12 for the precise statement. Proposition 2.4.12 proves that when investigating Serre type questions as above, it is enough to work in weight two. For example, in [22], some computational investigations of Serre's conjecture over imaginary quadratic fields were carried out and the principle illustrated by Proposition 2.4.12 was assumed to hold.

Here is our outline. We shall first recall Hecke theory in our context. This is the content of Section 2.2. In Section 2.3, we shall compare some modules. The main result is proved in Section 2.4.

2.2 Hecke operators

We define Hecke operators via Hecke correspondences on hyperbolic 3-manifolds. We start by fixing some notation. Let F be an imaginary quadratic field of class number $h \geq 1$. Denote by $\mathcal{O}$ its ring of integer and let $\mathfrak{n}$ be an ideal of $\mathcal{O}$. The class group of F is denoted by Cl and we fix a rational prime p inert in F and $\mathfrak{p} = p\mathcal{O}$. We also assume that $\mathfrak{p}$ is coprime with $\mathfrak{n}$. Let $\hat{\mathcal{O}}$ be the profinite completion of $\mathcal{O} : \hat{\mathcal{O}} = \prod_{\mathfrak{q} \neq 0} \mathcal{O}_{\mathfrak{q}}$. We will denote the adeles of F by $\mathbb{A}$, and $\mathbb{A}_f$, $\mathbb{A}_\infty$ stand for the finite part and the infinite part of $\mathbb{A}$. We write $G := \mathrm{GL}_2$, so that, $G(\mathbb{A}), G(F), G(\mathbb{A}_f)$ are the usual linear algebraic groups of 2×2 matrices with entries in $\mathbb{A}, F, \mathbb{A}_f$, respectively. Let $\mathbb{H}_3 := G(\mathbb{C})/\mathbb{C}^*\mathrm{U}_2 \cong \mathbb{C} \times \mathbb{R}_{>0}$, the three dimensional equivalent of the classical Poincaré upper half plane $\mathbb{H}_2 = G(\mathbb{R})/\mathbb{R}^*\mathrm{O}_2$. Here U_2 is the unitary subgroup of $G(\mathbb{C})$. Let K be an open compact subgroup of $G(\hat{\mathcal{O}})$

such that the determinant homomorphism

$$det : K \to \hat{\mathcal{O}}^*$$

is surjective. We define the following homogeneous space

$$\begin{aligned} Y_K &:= G(F)\backslash(\mathbb{H}_3 \times G(\mathbb{A}_f)/K) \\ &= G(F)\backslash(G(\mathbb{C})/\mathbb{C}^*\mathrm{U}_2 \times G(\mathbb{A}_f)/K) \\ &= G(F)\backslash G(\mathbb{A})/K.U_2.\mathbb{C}^*. \end{aligned}$$

By the determinant map we have

$$Y_K \twoheadrightarrow F^*\backslash\mathbb{A}^*/\hat{\mathcal{O}}^*\mathbb{C}^* \cong F^*\backslash\mathbb{A}_f^*/\hat{\mathcal{O}}^* \cong Cl.$$

2.2.1 Hecke correspondences and Hecke operators

Here we shall recall how a sheaf of $\overline{\mathbb{F}}_p$-modules on Y_K associated to a finite dimensional representation of $\overline{\mathbb{F}}_p[G(\mathbb{F}_{p^2})]$ is constructed. So, let σ be the generator of $\mathrm{Gal}(F/\mathbb{Q})$. Let

$$V_{\mathcal{O}} = V_{r,s}^{a,b}(\mathcal{O}) = \mathrm{Sym}^r(\mathcal{O}^2) \otimes det^a \otimes (\mathrm{Sym}^s(\mathcal{O}^2))^\sigma \otimes (det^b)^\sigma$$

be an $\mathcal{O}[G(\mathcal{O})]$-module endowed with the discrete topology. We define $V_{r,s}^{a,b}(\overline{\mathbb{F}}_p) := V_{\mathcal{O}} \otimes_{\mathcal{O}} \overline{\mathbb{F}}_p$. This space is also endowed with the discrete topology.

On the space $\mathbb{H}_3 \times G(\mathbb{A}_f) \times V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$, the group $G(F)$ acts on the first two factors from the left and the group K acts on the last two factors from the right. We write these double actions as follows. Let $(q,k) \in G(F) \times K$ and $(h,g,v) \in \mathbb{H}_3 \times G(\mathbb{A}_f) \times V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$ then:

$$(q,k) * (h,g,v) := (qh, qgk, k^{-1}.v).$$

Taking the quotients of these actions of $G(F)$, K on $\mathbb{H}_3 \times G(\mathbb{A}_f) \times V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$ yields a locally constant sheaf $\mathcal{V}_{\overline{\mathbb{F}}_p}$ of $\overline{\mathbb{F}}_p$-vector spaces associated to $V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$ on Y_K. More precisely let $X = G(\mathbb{A})/U_2\mathbb{C}^* \cong \mathbb{H}_3 \times G(\mathbb{A}_f)$. Under the assumption that K acts freely on $X \times V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$, one has a topological cover

$$\pi_1 : G(F)\backslash(X \times V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))/K \to G(F)\backslash X/K \cong Y_K.$$

And the locally constant sheaf $\mathcal{V}_{\overline{\mathbb{F}}_p}$ on Y_K is given by the sections of π_1 : for an open subset U of Y_K, we have

$$\mathcal{V}_{\overline{\mathbb{F}}_p}(U) = \{s : U \to G(F)\backslash(X \times V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))/K;\ \pi_1 \circ s = id\}.$$

Let $K' \subset K$ be another compact open subgroup of $G(\mathbb{A}_f)$. We have the natural projection $\psi : Y_{K'} \to Y_K$. We define the locally constant sheaf $\psi^{-1}\mathcal{V}_{\overline{\mathbb{F}}_p}$ of $\overline{\mathbb{F}}_p$-vector spaces on $Y_{K'}$ as the pull back of the sheaf $\mathcal{V}_{\overline{\mathbb{F}}_p}$. By functorial properties of sheaf cohomology the map ψ induces a homomorphism of $\overline{\mathbb{F}}_p$-vector spaces similar to the restriction homomorphism in group cohomology:

$$res : \mathrm{H}^r(Y_K, \mathcal{V}_{\overline{\mathbb{F}}_p}) \to \mathrm{H}^r(Y_{K'}, \psi^{-1}\mathcal{V}_{\overline{\mathbb{F}}_p}).$$

Since K' is a subgroup of finite index inside K, we have available the transfer map also known as the corestriction map :

$$cor : \mathrm{H}^r(Y_{K'}, \psi^{-1}\mathcal{V}_{\overline{\mathbb{F}}_p}) \to \mathrm{H}^r(Y_K, \mathcal{V}_{\overline{\mathbb{F}}_p}).$$

Next let $g \in Mat_2(\hat{\mathcal{O}})_{\neq 0}$ be such that all its local factors $g_{\mathfrak{q}}$ at almost all the finite places $\mathfrak{q}$ including those dividing $\mathfrak{pn}$ are $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and otherwise $g_{\mathfrak{q}}$ are of the form $\left(\begin{smallmatrix} \pi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \pi_{\mathfrak{q}}^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $\pi_{\mathfrak{q}}$ a uniformizer of $\mathcal{O}_{\mathfrak{q}}$.

Remark 2.2.1. *Often one takes $g \in Mat_2(\hat{\mathcal{O}})$ with the component at only one finite place $\mathfrak{q}$ away from $\mathfrak{pn}$ $g_{\mathfrak{q}}$ being of the form $\left(\begin{smallmatrix} \pi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and all the remaining components are the identity matrices.*

We define $K'_{g^{-1}} = K \cap g^{-1}Kg$ and $K'_g = gKg^{-1} \cap K$. The group isomorphism

$$K'_{g^{-1}} \cong K'_g;\ \lambda \mapsto g\lambda g^{-1}$$

induces the isomorphism $g^* : Y_{K'_{g^{-1}}} \cong Y_{K'_g}; y \mapsto gy$. We can now form the diagram

$$\begin{array}{ccc} Y_{K'_{g^{-1}}} & \xrightarrow{g^*} & Y_{K'_g} \\ \downarrow{\scriptstyle s_g} & & \downarrow{\scriptstyle \tilde{s}_g} \\ Y_K & & Y_K, \end{array}$$

where s_g and $\tilde{s}_g$ are the natural projections. This diagram is called a *Hecke correspondence* in light of the classical Hecke correspondence for modular curves. This picture is the essence of Hecke operators on cohomology, the notion of which we shall recall the definition in a moment. We denote by $s_g^{-1}\mathcal{V}_{\overline{\mathbb{F}}_p}$, $\tilde{s}_g^{-1}\mathcal{V}_{\overline{\mathbb{F}}_p}$ the sheaves on $Y_{K'_{g^{-1}}}$ and $Y_{K'_g}$ respectively, obtained as the pull back of the sheaf $\mathcal{V}_{\overline{\mathbb{F}}_p}$ of $\overline{\mathbb{F}}_p$-vector spaces on Y_K. Note that we have an isomorphism of sheaves induced by g^*, $conj_g : \tilde{s}_g^{-1}\mathcal{V}_{\overline{\mathbb{F}}_p} \cong s_g^{-1}\mathcal{V}_{\overline{\mathbb{F}}_p}$. Hence an isomorphism on cohomology

$$conj_g^* : \mathrm{H}^i(Y_{K'_{g^{-1}}}, s_g^{-1}\mathcal{V}_{\overline{\mathbb{F}}_p}) \cong \mathrm{H}^i(Y_{K'_g}, \tilde{s}_g^{-1}\mathcal{V}_{\overline{\mathbb{F}}_p})$$

holds. The Hecke operator T_g acting on the $\overline{\mathbb{F}}_p$-vector spaces $\mathrm{H}^i(Y_K, \mathcal{V}_{\overline{\mathbb{F}}_p})$ is defined by the following diagram:

$$\begin{array}{ccc}
\mathrm{H}^i(Y_{K'_{g^{-1}}}, s_g^{-1}\mathcal{V}_{\overline{\mathbb{F}}_p}) & \xrightarrow{conj_g^*} & \mathrm{H}^i(Y_{K'_g}, \tilde{s}_g^{-1}\mathcal{V}_{\overline{\mathbb{F}}_p}) \\
\uparrow{\scriptstyle res} & & \downarrow{\scriptstyle cor} \\
\mathrm{H}^i(Y_K, \mathcal{V}_{\overline{\mathbb{F}}_p}) & & \mathrm{H}^i(Y_K, \mathcal{V}_{\overline{\mathbb{F}}_p}).
\end{array}$$

So we have $T_g = cor \circ conj_g^* \circ res$. It is also known that T_g is independent of the choice of the uniformizers $\pi_{\mathfrak{q}}$ but in fact depends only on the double coset KgK. When $g \in Mat_2(\hat{\mathcal{O}})$ has local components $\begin{pmatrix} \pi_{\mathfrak{q}}^2 & 0 \\ 0 & 1 \end{pmatrix}$ at a finite number of finite places $\mathfrak{q}$ away from $\mathfrak{pn}$ and the identity otherwise, we shall denote the corresponding Hecke operator as S_g. For the sake of our understanding, let us translate the above diagram in group cohomology and have a more explicit description of the Hecke operator T_g.

2.2.2 Comparison with group cohomology and Hecke algebra

Let $\mathfrak{n}$ be a non-zero ideal of $\mathcal{O}$. For our purposes, we choose the following representatives of the class group Cl of F. By the Chebotarev density theorem, we can choose representatives of the class group $[\mathfrak{b}_1] = [\mathcal{O}], [\mathfrak{b}_2], \cdots, [\mathfrak{b}_h]$, where for $i > 1$, the $\mathfrak{b}_i$ are prime ideals coprime with $\mathfrak{pn}$. Thus we denote the class group as $Cl = \{[\mathfrak{b}_1], \cdots, [\mathfrak{b}_h]\}$. Let $\pi_{\mathfrak{b}_i}$ be a uniformizer of the local ring $\mathcal{O}_{\mathfrak{b}_i}$. We define $t_1 := (1, \cdots, 1, 1, 1, \cdots, 1, \cdots)$, and for $i > 1$, $t_i := (1, \cdots, 1, \pi_{\mathfrak{b}_i}, 1, \cdots, 1, \cdots) \in \mathbb{A}_f^*$, i.e, t_i is the idele having 1 at all places expect at the place $\mathfrak{b}_i$ where we have $\pi_{\mathfrak{b}_i}$. Via the group homomorphism

$$\begin{array}{rcl}
\mathbb{A}_f^* & \to & Cl \\
(\cdots x_{\mathfrak{q}} \cdots) & \mapsto & [\prod_{\mathfrak{q} \neq \infty} \mathfrak{q}^{v_{\mathfrak{q}}(x_{\mathfrak{q}})}],
\end{array}$$

where $v_{\mathfrak{q}}$ is the normalized valuation of $\mathcal{O}_{\mathfrak{q}}$, we see that t_i corresponds to $\mathfrak{b}_i$. We define $g_i := \begin{pmatrix} t_i & 0 \\ 0 & 1 \end{pmatrix}$, i.e, $(g_i)_{\mathfrak{q}} = \begin{pmatrix} (t_i)_{\mathfrak{q}} & 0 \\ 0 & 1 \end{pmatrix}$. Similarly g_i corresponds to the class $[\mathfrak{b}_i]$ via the determinant map.

From the strong approximation theorem, the topological space Y_K decomposes into the disjoint union of its connected components as:

$$Y_K = \amalg_{i=1}^{h} \Gamma_{[\mathfrak{b}_i]} \backslash \mathbb{H}_3,$$

where $\Gamma_{[\mathfrak{b}_i]} := G(F) \cap g_i K g_i^{-1}$. This is an arithmetic subgroup of $G(F)$. We next recall the definition of neatness for subgroups of $G(\mathbb{A}_f)$. This is the condition to ensure that K acts freely on $X \times V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$, where X and $V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$ are defined in Subsection 2.2.1.

Neatness

Let $\overline{F}$ be an algebraic closure of F. A subgroup Γ of $G(F)$ is said to be *neat* if and only if for all $g \in \Gamma$, the multiplicative subgroup of $\overline{F}^*$ generated by all the eigenvalues of g is torsion free. If Γ is neat then it is torsion free. Let $\mathfrak{q}$ be a finite place of F and consider $F_{\mathfrak{q}}$. We fix an embedding $\overline{F} \hookrightarrow \overline{F}_{\mathfrak{q}}$. Let $g = (g_{\mathfrak{q}}) \in G(\mathbb{A}_f)$ and let $\Omega_{\mathfrak{q}}$ be the subgroup of $\overline{F}_{\mathfrak{q}}^*$ generated by all the eigenvalues of $g_{\mathfrak{q}}$. One says that g is neat if only if we have

$$\bigcap_{\mathfrak{q}} (\overline{F}^* \cap \Omega_{\mathfrak{q}})_{tor} = \{1\}.$$

A subgroup Ω of $G(\mathbb{A}_f)$ is neat if and only if all its elements are neat. For $g \in G(\hat{\mathcal{O}})$, if the open compact subgroup K of $G(\hat{\mathcal{O}})$ is neat then $G(F) \cap gKg^{-1}$ is also neat. For more on the neatness condition see Borel [7, p. 117].

So we choose K to be neat so that the groups $\Gamma_{[\mathfrak{b}_i]}$ are torsion free. To achieve this, if $K = K_1(\mathfrak{n})$, the open compact subgroup of level $\mathfrak{n}$ defined below, where the positive generator of $\mathfrak{n} \cap \mathbb{Z}$ is greater than 3, then $\Gamma_{[\mathfrak{b}_i]}$ are torsion free. This is Lemma 2.3.1 from [37]. This being given, the smooth manifolds $\Gamma_{[\mathfrak{b}_i]} \backslash \mathbb{H}_3$ are Eilenberg-McLane spaces of type $K(\Gamma_{[\mathfrak{b}_i]}, 1)$ (this K has nothing to do with our open compact subgroup K, this is just an unfortunate clash between two pieces of standard notation), i.e, $\pi_1(\Gamma_{[\mathfrak{b}_i]} \backslash \mathbb{H}_3) = \Gamma_{[\mathfrak{b}_i]}$ and $\pi_n(\Gamma_{[\mathfrak{b}_i]} \backslash \mathbb{H}_3) = 1$ for $n > 1$. From a general comparison theorem it is known that an isomorphism $\mathrm{H}^r(\Gamma_{[\mathfrak{b}_i]} \backslash \mathbb{H}_3, \mathcal{V}_{\overline{\mathbb{F}}_p}) = \mathrm{H}^r(\Gamma_{[\mathfrak{b}_i]}, V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$ holds, see [9] for details. Hence we can write

$$\mathrm{H}^r(Y_K, \mathcal{V}_{\overline{\mathbb{F}}_p}) = \oplus_{i=1}^{h} \mathrm{H}^r(\Gamma_{[\mathfrak{b}_i]} \backslash \mathbb{H}_3, \mathcal{V}_{\overline{\mathbb{F}}_p}) = \oplus_{i=1}^{h} \mathrm{H}^r(\Gamma_{[\mathfrak{b}_i]}, V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)).$$

Let us further specialize the open compact subgroup K. We define the open compact subgroup of level $\mathfrak{n}$

$$K_1(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \prod_{\mathfrak{q}\nmid\infty} G(\mathcal{O}_\mathfrak{q}) : c, d-1 \in \mathfrak{n}\hat{\mathcal{O}} \right\}.$$

This is an open compact subgroup which surjects on $\hat{\mathcal{O}}^*$ by the determinant map. The corresponding congruence subgroups $G(F) \cap g_i K_1(\mathfrak{n}) g_i^{-1}$ are denoted as $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. As already alluded to, the Hecke operators T_g do not act componentwise on the $\overline{\mathbb{F}}_p$-vector space $\oplus_{i=1}^h \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}, V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$. By this we mean that in general T_g permutes the components when acting on an element from $\oplus_{i=1}^h \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}, V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$ as we will soon see.

Some formulas for the Hecke action

We recall here the formulas defining the Hecke action on group cohomology. To this end, let us first introduce some more notation. Let $\mathfrak{q}$ be an integral ideal away from $\mathfrak{p}\mathfrak{n}$. We consider the following subset of $Mat_2(\hat{\mathcal{O}})$. Define

$$\Delta_1^\mathfrak{q}(\mathfrak{n}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Mat_2(\hat{\mathcal{O}}) : (ad-bc)\hat{\mathcal{O}} = \mathfrak{q}\hat{\mathcal{O}}, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \pmod{\mathfrak{n}} \}.$$

The open compact subgroup $K_1(\mathfrak{n})$ acts on $\Delta_1^\mathfrak{q}(\mathfrak{n})$ via multiplication: for $g \in K_1(\mathfrak{n})$ and $\delta \in \Delta_1^\mathfrak{q}(\mathfrak{n})$ we have $g\delta \in \Delta_1^\mathfrak{q}(\mathfrak{n})$. We have that $\Delta_1^\mathfrak{q}(\mathfrak{n}) K_1(\mathfrak{n}) = K_1(\mathfrak{n}) \Delta_1^\mathfrak{q}(\mathfrak{n}) = \Delta_1^\mathfrak{q}(\mathfrak{n})$. For $\delta \in \Delta_1^\mathfrak{q}(\mathfrak{n})$ we define the subgroup

$$K'_{1,\delta}(\mathfrak{n}) = \delta K_1(\mathfrak{n}) \delta^{-1} \cap K_1(\mathfrak{n})$$

of $K_1(\mathfrak{n})$. The subsets $\Delta_1^\mathfrak{q}(\mathfrak{n})$ act on any left $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathbb{F}_{p^2})]$-module via reduction modulo $\mathfrak{p}$. There is the following fact that is worth mentioning.

Lemma 2.2.2. *Let $\delta \in \Delta_1^\mathfrak{q}(\mathfrak{n})$. Then there is a bijection between the coset space $K_1(\mathfrak{n})/K'_{1,\delta}(\mathfrak{n})$ and the orbit space $K_1(\mathfrak{n})\delta K_1(\mathfrak{n})/K_1(\mathfrak{n})$ given as*

$$\begin{aligned} K_1(\mathfrak{n})/K'_{1,\delta}(\mathfrak{n}) &\to K_1(\mathfrak{n})\delta K_1(\mathfrak{n})/K_1(\mathfrak{n}) \\ \lambda K'_{1,\delta}(\mathfrak{n}) &\mapsto \lambda\delta K_1(\mathfrak{n}). \end{aligned}$$

Proof. There is a surjective map $K_1(\mathfrak{n}) \to K_1(\mathfrak{n})\delta K_1(\mathfrak{n})/K_1(\mathfrak{n})$ which sends $\lambda K'_{1,\delta}(\mathfrak{n})$ to $\lambda\delta K_1(\mathfrak{n})$. Two distinct elements λ and λ' map to the same orbit if and only of they lie in the same class modulo $K'_{1,\delta}(\mathfrak{n})$. □

For $\delta \in \Delta_1^\mathfrak{q}(\mathfrak{n})$, there are finitely many $\gamma_j \in \Delta_1^\mathfrak{q}(\mathfrak{n})$ such that the double coset

$K_1(\mathfrak{n})\delta K_1(\mathfrak{n})$ decomposes as

$$K_1(\mathfrak{n})\delta K_1(\mathfrak{n}) = \amalg_j \gamma_j K_1(\mathfrak{n}).$$

Let $g \in Mat_2(\hat{\mathcal{O}})$ be such that its components at a finite number of finite places $\mathfrak{q}$ away from $\mathfrak{pn}$ are of the form $\left(\begin{smallmatrix} \pi_{\mathfrak{q}} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \pi_{\mathfrak{q}}^2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $\pi_{\mathfrak{q}}$ is a uniformizer of $\mathcal{O}_{\mathfrak{q}}$ and are the identity otherwise. When we denote $\mathfrak{c} = (det(g))$ the ideal corresponding to g, then $g \in \Delta_1^{\mathfrak{c}}(\mathfrak{n})$.

Lemma 2.2.3. *Let $g \in \Delta_1^{\mathfrak{c}}(\mathfrak{n})$ as above. Let g_i corresponding to $[\mathfrak{b}_i]$ and $K_1(\mathfrak{n})$ as above. Then, for each i there exist a unique index j_i, $1 \leq j_i \leq h$, matrices $k_i = \left(\begin{smallmatrix} u_i & 0 \\ 0 & 1 \end{smallmatrix}\right) \in g_i K_1(\mathfrak{n}) g_i^{-1}$ and $\beta_i := g_{j_i} g g_i^{-1} k_i = \left(\begin{smallmatrix} y_i & 0 \\ 0 & 1 \end{smallmatrix}\right) \in G(F)$ such that $K_1(\mathfrak{n}) g K_1(\mathfrak{n}) = K_1(\mathfrak{n}) g_{j_i}^{-1} \beta_i g_i K_1(\mathfrak{n})$.*

Proof. For each i let j_i be the unique index such that the ideal $(det(g_{j_i} g g_i^{-1}))$ is principal. Set then $\alpha_i := g_{j_i} g g_i^{-1} = \left(\begin{smallmatrix} det(\alpha_i) & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The ideal $(det(\alpha_i))$ being principal means that $det(\alpha_i) = x_i y_i$ with $y_i \in F^*$ and $x_i \in \hat{\mathcal{O}}^*$. Set $u_i = x_i^{-1}$ and define $k_i = \left(\begin{smallmatrix} u_i & 0 \\ 0 & 1 \end{smallmatrix}\right) \in K_1(\mathfrak{n})$. Then $k_i \in g_i K_1(\mathfrak{n}) g_i^{-1}$ and $\beta_i := \alpha_i k_i = \left(\begin{smallmatrix} y_i & 0 \\ 0 & 1 \end{smallmatrix}\right) \in G(F)$. Hence for each i there exists a matrix

$$\beta_i \in g_{j_i} \Delta_1^{\mathfrak{c}}(\mathfrak{n}) K_1(\mathfrak{n}) g_i^{-1} \cap G(F) = g_{j_i} \Delta_1^{\mathfrak{c}}(\mathfrak{n}) g_i^{-1} \cap G(F)$$

such that $K_1(\mathfrak{n}) g K_1(\mathfrak{n}) = K_1(\mathfrak{n}) g_{j_i}^{-1} \beta_i g_i K_1(\mathfrak{n})$. Indeed, $g_{j_i}^{-1} \beta_i g_i = g_{j_i}^{-1} \alpha_i k_i g_i = g g_i^{-1} k_i g_i$, and we observe that we have $g_i^{-1} k_i g_i \in K_1(\mathfrak{n})$. $\square$

For $1 \leq i \leq h$, let j_i and β_i as given in the above lemma. Let $\mathfrak{f}_i := (det(\beta_i)) = \mathfrak{b}_{j_i} \mathfrak{b}_i^{-1} \mathfrak{c}$. Define $\Lambda_{1,[\mathfrak{b}_i]}^{\mathfrak{f}_i}(\mathfrak{n}) := g_{j_i} \Delta_1^{\mathfrak{c}}(\mathfrak{n}) g_i^{-1} \cap G(F)$. Explicitly this is the set

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(F) : a \in \mathfrak{b}_{j_i} \mathfrak{b}_i^{-1}, b \in \mathfrak{b}_{j_i}, c \in \mathfrak{b}_i^{-1}, d - 1 \in \mathfrak{n}\mathcal{O}; (ad - bc)\mathcal{O} = \mathfrak{f}_i \right\}.$$

We set $j := j_i$. Let $\alpha \in \Lambda_{1,[\mathfrak{b}_i]}^{\mathfrak{f}_i}(\mathfrak{n})$ (we have in mind β_i). We consider the double coset $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}) \alpha \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. This double coset defines a Hecke operator T_α mapping

$$\mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)) \text{ to } \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$$

as follows. Firstly one needs to introduce the following subgroups

1. $\Gamma_{1,[\mathfrak{b}_i]}^{\prime,\alpha^{-1}}(\mathfrak{n}) := \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \cap \alpha^{-1} \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}) \alpha$

2. $\Gamma_{1,[\mathfrak{b}_j]}^{\prime\prime,\alpha}(\mathfrak{n}) := \alpha \Gamma'_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \alpha^{-1} = \alpha \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \alpha^{-1} \cap \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})$.

The operator T_α is defined as the composition of the following maps:

$$\begin{array}{ccc} \mathrm{H}^r(\Gamma'^{,\alpha^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) & \xrightarrow{conj_\alpha} & \mathrm{H}^r(\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \\ \uparrow{\scriptstyle res} & & \downarrow{\scriptstyle cor} \\ \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) & & \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)). \end{array}$$

Here *res* is the restriction map, $conj_\alpha$ is the isomorphism induced by the compatible maps:

$$\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n}) \cong \Gamma'^{,\alpha^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n})$$
$$\omega \mapsto \alpha^{-1}\omega\alpha$$

and

$$V^{a,b}_{r,s}(\overline{\mathbb{F}}_p) \to V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)$$
$$v \mapsto \alpha.v.$$

Here *cor* is the corestriction homomorphism. We explicitly describe T_α in degree zero and one. In degree zero T_α is given as

$$\begin{array}{ccc} \mathrm{H}^0(\Gamma'^{,\alpha^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) & \xrightarrow{v\mapsto\alpha v} & \mathrm{H}^0(\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \\ \uparrow{\scriptstyle v\mapsto v} & & \downarrow{\scriptstyle v\mapsto\sum_h hv} \\ \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) & & \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)). \end{array}$$

where the sum is over a set of cosets representatives of $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})/\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})$. Hence one obtains that:

$$\begin{array}{rcl} T_\alpha : \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) & \to & \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \\ v & \mapsto & \displaystyle\sum_{\lambda\in\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})/\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})} (\lambda\alpha).v. \end{array}$$

It is worthwhile observing that the decomposition $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}) = \amalg_r \lambda_r \Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})$ is equivalent to the decomposition of the double cosets $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})\alpha\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = \amalg_r \lambda_r \alpha \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$.

Formula on degree one

We now give the formula of T_α on degree one cohomology. To this end, we need to recall the formulas describing the isomorphism $conj_\alpha$ and the corestriction in terms of non-homogeneous cocycles. The conjugation isomorphism is described by the formula

$$\begin{aligned} conj_\alpha : \mathrm{H}^1(\Gamma'^{,\alpha^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) &\rightarrow \mathrm{H}^1(\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \\ c &\mapsto (\omega \mapsto \alpha.c(\alpha^{-1}\omega\alpha)). \end{aligned}$$

For the corestriction homomorphism, let $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}) = \amalg_n \gamma_n \Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})$. For $\omega \in \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})$, let s_n be the unique index such that $\gamma_n^{-1}\omega\gamma_{s_n} \in \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})$. Then the corestriction homomorphism is given as

$$\begin{aligned} cor : \mathrm{H}^1(\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) &\rightarrow \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \\ c &\mapsto (\omega \mapsto \sum_n \gamma_n.c(\gamma_n^{-1}\omega\gamma_{s_n})). \end{aligned}$$

The formula of T_α on degree one cohomology is thus

$$\begin{aligned} T_\alpha : \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) &\rightarrow \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}), V^{a,b}_{r,s}(\overline{\mathbb{F}}_p)) \\ c &\mapsto (\omega \mapsto \sum_n \gamma_n\alpha.c((\gamma_n\alpha)^{-1}\omega\gamma_{s_n}\alpha)). \end{aligned}$$

Indeed with the given formulas we have

$$\begin{aligned} (cor(conj_\alpha(c)))(w) &= \sum_{\gamma_n \in \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})/\Gamma'_{1,[\mathfrak{b}_j]}(\mathfrak{n})} \gamma_n.(conj_\alpha(c)(\gamma_n^{-1}w\gamma_{s_n})) \\ &= \sum_{\gamma_n \in \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})/\Gamma'_{1,[\mathfrak{b}_j]}(\mathfrak{n})} \gamma_n\alpha.c(\alpha^{-1}\gamma_n^{-1}w\gamma_{s_n}\alpha). \end{aligned}$$

Let λ_i be another set of representatives of $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})/\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})$, and $\sigma_i \in \Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}$ such that $\lambda_i = \gamma_i\sigma_i$. With this we have

$$(cor(c))(w) = \gamma_i\sigma.c(\sigma_i^{-1}\gamma_i^{-1}w\gamma_{j_i}\sigma_i).$$

Because taking conjugation by an element from $\Gamma'^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})$ gives cohomologous cocycle, we deduce that the corestriction map does not depend on the choice of representatives of $\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})/\Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]}(\mathfrak{n})$. This means that T_α does not depend on the choice of set of representatives and so only depends on the double coset $\Gamma_{1,[\mathfrak{b}_j]}\alpha\Gamma_{1,[\mathfrak{b}_i]}$ since we know that

$$\Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n}) = \amalg_n \gamma_n \Gamma''^{,\alpha}_{1,[\mathfrak{b}_j]} \Longleftrightarrow \Gamma_{1,[\mathfrak{b}_j]}(\mathfrak{n})\alpha\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = \amalg_n \gamma_n \alpha \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}).$$

Action of T_g on $\oplus_{i=1}^h \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$

Now that we have recalled the formulas of the Hecke operators on group cohomology, let us say how Hecke operators act on the $\overline{\mathbb{F}}_p$-vector spaces $\oplus_{i=1}^h \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$. Let g be as in Lemma 2.2.3 and consider β_i and j_i provided by the lemma loc. cit. Let T_{β_i} the Hecke operator corresponding to the double coset $\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. Then T_{β_i} sends an element from $\mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$ to $\mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$. It was proved by Shimura, see [33], that for $(c_1, \cdots, c_h) \in \oplus_{i=1}^h \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$, the Hecke action of T_g is

$$T_g.(c_1, \cdots, c_h) = (d_1, \cdots .d_h),$$

where $d_{j_i} = T_{\beta_i}.c_i$.

Remark 2.2.4. *In the idyllic situation where the ideal $(det(g))$ is principal, then, the Hecke operator T_g does not permute the summands in $\oplus_{l=1}^h \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_l]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$. Indeed $(det(g_{j_i} g g_i^{-1})) = (det(g))$, so $j_i = i$ in Lemma 2.2.3. Therefore $T_g.(c_1, \cdots, c_h) = (d_1, \cdots, d_h)$ where $d_i = T_{\beta_i}.c_i$.*

Remark 2.2.5. *Let g be as in Lemma 2.2.3. Let us denote the ideal $(det(g))$ as $\mathfrak{c}$. Then T_g maps the $\overline{\mathbb{F}}_p$-vector spaces $\oplus_{l=1}^r \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_l]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$ to $\oplus_{l=1}^r \mathrm{H}^r(\Gamma_{1,[\mathfrak{c}^{-1}\mathfrak{b}_l]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$. To see this, one needs to just recall that T_g maps*

$$\oplus_{i=1}^r \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)) \text{ to } \oplus_{i=1}^r \mathrm{H}^r(\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$$

where j_i is such that $(det(g_{j_i} g g_i^{-1}))$ is principal. In terms of ideals this means that $[\mathfrak{c}^{-1}\mathfrak{b}_i] = [\mathfrak{b}_{j_i}]$.

We shall next recall a definition of a class of degree one Hecke operators known as *diamond operators.*

Diamond operators

This is a special kind (degree one Hecke operator) of Hecke operator defined as follows. Define the open compact subgroup $K_0(\mathfrak{n})$ of $G(\hat{\mathcal{O}})$ as

$$K_0(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \prod_{\mathfrak{q}\nmid\infty} G(\mathcal{O}_\mathfrak{q}) : c \in \mathfrak{n}\hat{\mathcal{O}} \right\}.$$

Then $K_1(\mathfrak{n})$ is a normal subgroup of $K_0(\mathfrak{n})$. So for any $\alpha \in K_0(\mathfrak{n})$ we have

$$\alpha K_1(\mathfrak{n})\alpha^{-1} = K_1(\mathfrak{n}).$$

Therefore we deduce that $K_1(\mathfrak{n})\alpha K_1(\mathfrak{n}) = \alpha K_1(\mathfrak{n})$. The Hecke operator corresponding to the double coset $K_1(\mathfrak{n})\alpha K_1(\mathfrak{n})$ is called a *diamond operator.*

Example 2.2.6. *Take $\alpha \in K_0(\mathfrak{n})$ with determinant corresponding to a principal ideal such that at one place $\mathfrak{q}$ dividing $\mathfrak{n}$ the component $\alpha_{\mathfrak{q}}$ has reduction modulo $\mathfrak{n}$ a matrix of the form $\left(\begin{smallmatrix} \omega & 0 \\ 0 & \gamma \end{smallmatrix}\right)$ and at the other places the components are the identity matrix. Because the determinant of α is principal, the Hecke operator T_α does not permute the components:*

$$\begin{aligned} T_\alpha : \oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)) &\rightarrow \oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)) \\ (c_1, \cdots, c_h) &\mapsto (T_{\beta_1}.c_1, \cdots, T_{\beta_h}.c_h) \end{aligned}$$

where $\beta_i \in \Gamma_{0,[\mathfrak{b}_i]}(\mathfrak{n}) := g_i K_0(\mathfrak{n}) g_i^{-1} \cap G(F)$ such that $\alpha K_1(\mathfrak{n}) = g_i^{-1}\beta_i g_i K_1(\mathfrak{n})$ and T_{β_i} is the Hecke operator corresponding to the double coset $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. Note that T_{β_i} defines a non-adelic diamond operator. More explicitly the Hecke operator T_{β_i} on $\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$ is given as

$$\begin{aligned} T_{\beta_i} : \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)) &\rightarrow \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)) \\ c &\mapsto (w \mapsto \beta_i.c(\beta_i^{-1} w \beta_i)). \end{aligned}$$

Aside from this interesting fact, there is a nice interpretation of diamond operators as in the classical setting. It arises from the isomorphism of abelian groups

$$\begin{aligned} K_0(\mathfrak{n})/K_1(\mathfrak{n}) &\rightarrow (\hat{\mathcal{O}}/\mathfrak{n}\hat{\mathcal{O}})^* \cong (\mathcal{O}/\mathfrak{n})^* \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) &\mapsto d \pmod{\mathfrak{n}}. \end{aligned}$$

This means that we have an action of the group $(\mathcal{O}/\mathfrak{n})^*$ on $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p))$. Let $\chi : (\mathcal{O}/\mathfrak{n})^* \rightarrow \overline{\mathbb{F}}_p^*$ be a character. As a representation of the abelian group $(\mathcal{O}/\mathfrak{n})^*$, then when $p \nmid \sharp(\mathcal{O}/\mathfrak{n})^*$, the space $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$ decomposes as a direct sum of χ-eigenspaces. So by denoting the spaces $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$ as $\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n})$ and a χ-eigenspace as $\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n}, \chi)$, then we have

$$\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n}) = \oplus_\chi \mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n}, \chi).$$

Let us turn next to the definition of the Hecke algebra.

Hecke algebra

We start by defining first what we call mod p cohomological modular forms over F. Recall that we have denoted $p\mathcal{O}$ as $\mathfrak{p}$ and we are assuming that p is inert in F. The residue field is then $\mathbb{F}_{p^2}$. The congruence subgroups $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ act on $V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$ via reduction modulo p.

Definition 2.2.7. *A cohomological mod p modular form of weight $V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$ and level $\mathfrak{n}$ over F is a class in*

$$\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)).$$

We have denoted this $\overline{\mathbb{F}}_p$-vector spaces as $\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n})$.

We next define the Hecke algebra of interest for our purposes.

Definition 2.2.8 (Hecke algebra)**.** *1. The abstract Hecke algebra $\mathcal{H}$ is the polynomial algebra $\mathbb{Z}[T_{\mathfrak{q}}, S_{\mathfrak{q}} |\ \mathfrak{q} \nmid \mathfrak{pn}$ maximal ideal $\subset \mathcal{O}]$.*

2. The Hecke algebra $\mathcal{H}(\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n}))$ acting on $\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n})$ is the homomorphic image of: $\mathcal{H} \to End_{\overline{\mathbb{F}}_p}(\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n})); T_{\mathfrak{q}}, S_{\mathfrak{q}} \mapsto T_{\mathfrak{q}}, S_{\mathfrak{q}}$.

As we said an eigenform for all the Hecke operator $T_{\mathfrak{q}}$ for $\mathfrak{q}$ away from $\mathfrak{pn}$ gives rise to a system of Hecke eigenvalues. Here is a formal definition of a system of Hecke eigenvalues with values in $\overline{\mathbb{F}}_p$.

Definition 2.2.9. *A system of Hecke eigenvalues with values in $\overline{\mathbb{F}}_p$ is a ring homomorphism $\psi : \mathcal{H} \to \overline{\mathbb{F}}_p$. We say that it occurs in $\mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n})$ if there is a non-zero $f \in \mathcal{M}_{V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)}(\mathfrak{n})$ such that $Tf = \psi(T)f$ for all $T \in \mathcal{H}$.*

In the next section we shall relate the induced modules $\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$ to some irreducible $\overline{\mathbb{F}}_p[\mathrm{GL}_2(\mathcal{O})]$-modules of the form $V_{r,s}^{a,b}(\overline{\mathbb{F}}_p)$.

2.3 The relevant induced modules

We recall that by assumption we have fixed a rational inert prime p and $\mathfrak{p} = p\mathcal{O}$ does not divide an integral ideal $\mathfrak{n}$ which was also fixed. Here we will be concerned with the induced modules $\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$. We shall derive a more explicit decomposition of the latter. Let $\tilde{G} = \mathrm{GL}_2(\mathbb{F}_{p^2})$ and $\tilde{S} = \mathrm{SL}_2(\mathbb{F}_{p^2})$.

Define the following congruence subgroups of $\mathrm{GL}_2(F)$:

$$\Gamma_{1,[\mathfrak{b}_i]}^1(\mathfrak{n}) := g_i K_1(\mathfrak{n}) g_i^{-1} \cap \mathrm{SL}_2(F).$$

Because $\left(\begin{smallmatrix} t_i & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\begin{pmatrix} t_i^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & t_i b \\ t_i^{-1}c & d \end{pmatrix}$, one obtains that

$$\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(F) : a-1, d-1 \in \mathfrak{n}; b \in \mathfrak{b}_i, c \in \mathfrak{b}_i^{-1}\mathfrak{n}\right\}.$$

In particular with our assumptions one has that

$$\Gamma^1_{1,[\mathfrak{b}_1]}(\mathfrak{n}) = \Gamma_1(\mathfrak{n}) := \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}) : a-1,\ d-1,\ c \equiv 0 \pmod{\mathfrak{n}}\right\}.$$

Furthermore, let

$$\Gamma(\mathfrak{n}) = \left\{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathcal{O}) : a-1,\ d-1,\ b,\ c \equiv 0 \pmod{\mathfrak{n}}\right\}.$$

Lemma 2.3.1. *Let $\tilde{S} = \mathrm{SL}_2(\mathbb{F}_{p^2})$. Then, we have an exact sequence*

$$1 \to \Gamma(\mathfrak{p}) \cap \Gamma_1(\mathfrak{n}) \to \Gamma_1(\mathfrak{n}) \to \tilde{S} \to 1$$

where the third arrow is reduction modulo p.

Proof. It is clear that $\Gamma(\mathfrak{p}) \cap \Gamma_1(\mathfrak{n})$ is the kernel of the reduction modulo $\mathfrak{p}$ of $\Gamma_1(\mathfrak{n})$. So, we are left to see the surjectivity of the third arrow. To this end let $a, b, c, d \in \mathcal{O}$ with $ad - bc \equiv 1 \pmod{\mathfrak{p}}$. We need to find $\alpha, \beta, \gamma, \delta \in \mathcal{O}$ such that $\alpha\delta - \beta\gamma = 1$ with the congruences:

$$\begin{array}{rcll} \alpha & \equiv & a & \pmod{\mathfrak{p}} \\ \alpha & \equiv & 1 & \pmod{\mathfrak{n}} \\ \beta & \equiv & b & \pmod{\mathfrak{p}} \\ \gamma & \equiv & c & \pmod{\mathfrak{p}} \\ \gamma & \equiv & 0 & \pmod{\mathfrak{n}} \\ \delta & \equiv & d & \pmod{\mathfrak{p}} \\ \delta & \equiv & 1 & \pmod{\mathfrak{n}}. \end{array}$$

It is readily seen that if $0 \neq c \in \mathfrak{n}$ and is coprime with $\mathfrak{p}$ then the Chinese Remainder Theorem permits to conclude. Indeed, set $\gamma = c$, there exist $\alpha, \delta \in \mathcal{O}$ with $\alpha \equiv a \pmod{\mathfrak{p}}$, $\alpha \equiv 1 \pmod{\gamma}$, $\delta \equiv d \pmod{\mathfrak{p}}$, $\delta \equiv 1 \pmod{\gamma}$. This gives $\alpha\delta \equiv 1 \pmod{\gamma}$, and so there exists $\beta \in \mathcal{O}$ such that $\alpha\delta - \beta\gamma = 1$ and $\beta \equiv b \pmod{\mathfrak{p}}$. So we need to see that we can always reduce to this case. To this end as $\mathfrak{n}$ is coprime with $\mathfrak{p}$, we can find $n \in \mathfrak{n}$, $k \in \mathfrak{p}$, and $r, s \in \mathcal{O}$ such that $nr - ks = 1$. The image of the matrix $\left(\begin{smallmatrix} r & 0 \\ n & n \end{smallmatrix}\right)$ belongs to $\tilde{S}$ and can be lifted by the previous arguments. Then $\left(\begin{smallmatrix} r & 0 \\ n & n \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} ra & rb \\ n(c+a) & n(d+b) \end{smallmatrix}\right)$ is a matrix in $\tilde{S}$ whose bottom line has entries in $\mathfrak{n}$. Then if $n(c+a) \neq 0$ we are done, otherwise we just have to multiply from the right by $\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right)$ for the condition to hold. □

Corollary 2.3.2. *For $i \geq 1$, the congruence subgroup $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ surjects onto $\tilde{S}$ via reduction modulo p.*

Proof. From Lemma 2.3.1, we have that $\Gamma_1(\mathfrak{n})$ surjects onto $\tilde{S}$. So let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \tilde{S}$ and $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \Gamma_1(\mathfrak{n})$ be a lift of M. For each $i > 1$ take $\lambda_i \in \mathfrak{b}_i$ such that $\lambda_i \equiv 1 \pmod{\mathfrak{p}}$ (this is possible since $\mathfrak{b}_i$ is coprime with $\mathfrak{p}$). Then the matrix $\left(\begin{smallmatrix} \alpha & \lambda_i\beta \\ \lambda_i^{-1}\gamma & \delta \end{smallmatrix}\right)$ belongs to $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ and it has reduction M. □

From the fact that $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \subset \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$, we deduce that the reduction modulo p of $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ contains $\tilde{S}$. Now suppose we are given two subgroups H_1, H_2 of $\tilde{G} = \mathrm{GL}_2(\mathbb{F}_{p^2})$ containing $\tilde{S}$ and such that their images by the determinant map are the same: $det(H_1) = det(H_2) < \mathbb{F}^*_{p^2}$. The fact $det(H_1 \cap H_2) = det(H_1) \cap det(H_2)$ implies the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \tilde{S} & \longrightarrow & H_1 \cap H_2 & \xrightarrow{det} & det(H_1) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \tilde{S} & \longrightarrow & H_1 & \xrightarrow{det} & det(H_1) & \longrightarrow & 1.
\end{array}$$

Therefore one has $H_1 = H_2$, and we have established that any subgroup H of $\tilde{G}$ containing $\tilde{S}$ is uniquely determined by the image of the determinant map $H \xrightarrow{det} \mathbb{F}^*_{p^2}$. From this fact we derive that $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ reduces to

$$\mathcal{T}_1(\mathfrak{n}) := \left\{ g \in \tilde{G} : det(g) \in Im(\mathcal{O}^* \xrightarrow{reduction} \mathbb{F}^*_{p^2}) \right\}.$$

We also derive that $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})$ reduces to

$$\mathcal{T}_1(\mathfrak{p}\mathfrak{n}) := \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in \tilde{G} : a \in Im(\mathcal{O}^* \xrightarrow{reduction} \mathbb{F}^*_{p^2}) \right\}.$$

In summary, reduction mod p gives the following bijection:

$$\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}) \backslash \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \to \mathcal{T}_1(\mathfrak{p}\mathfrak{n}) \backslash \mathcal{T}_1(\mathfrak{n}).$$

Define $\tilde{U} = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in \tilde{G} \right\}$. Next we have the following bijection

$$\begin{array}{rcl} \mathcal{T}_1(\mathfrak{pn})\backslash\mathcal{T}_1(\mathfrak{n}) & \longleftrightarrow & \tilde{U}\backslash\tilde{G} \\ \mathcal{T}_1(\mathfrak{pn})g & \mapsto & \tilde{U}g. \end{array}$$

Indeed the map is surjective and two elements from $\mathcal{T}_1(\mathfrak{n})$ are sent to the same class modulo $\tilde{U}$ if and only they belong to the same class modulo $\mathcal{T}_1(\mathfrak{pn})$ because we have $\tilde{U} \cap \mathcal{T}_1(\mathfrak{n}) = \mathcal{T}_1(\mathfrak{pn})$. Composing these two bijections, we obtain the bijection

$$\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})\backslash\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \longleftrightarrow \tilde{U}\backslash\tilde{G}.$$

Induced modules

Let H be a group and $J < H$ a subgroup of finite index. For a left J-module M the induced module, and a twisted induced module are defined as follows.

Definition 2.3.3. *1.* $\mathrm{Ind}_J^H(M) = \{f : H \to M : f(gh) = gf(h)\ \forall\ g \in J, h \in H\}$.

2. Given a character $\chi : J \to \overline{\mathbb{F}}_p^*$, we define a twisted induced module as

$$\mathrm{Ind}_J^H(\overline{\mathbb{F}}_p^\chi) = \{f : H \to \overline{\mathbb{F}}_p : f(gh) = \chi(g)f(h)\ \forall\ g \in J, h \in H\}.$$

Recall how a left action of H on $\mathrm{Ind}_J^H(M)$ can be defined: for $g \in H$ and $f \in \mathrm{Ind}_J^H(M)$ we have $(g.f)(h) := f(hg)$.

Let $\tilde{B} = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & e \end{smallmatrix}\right) \in \tilde{G} \right\}$ be the Borel subgroup of $\tilde{G}$ and define the character χ of $\tilde{B}$ by

$$\begin{array}{rcl} \chi : \tilde{B} & \to & \mathbb{F}_{p^2}^* \\ \left(\begin{smallmatrix} a & b \\ 0 & e \end{smallmatrix}\right) & \mapsto & e. \end{array}$$

For an integer d, we also set $\chi^d(.) = (\chi(.))^d$. The homomorphism χ induces a group isomorphism

$$\tilde{U}\backslash\tilde{B} \cong \mathbb{F}_{p^2}^*.$$

From this isomorphism we obtain the following isomorphism of $\tilde{B}$-modules

$$\mathrm{Ind}_{\tilde{U}}^{\tilde{B}}(\overline{\mathbb{F}}_p) \cong \mathrm{Ind}_{\{1\}}^{\mathbb{F}_{p^2}^*}(\overline{\mathbb{F}}_p).$$

The isomorphism is defined as follows:

$$\begin{array}{rcl}\Phi : \mathrm{Ind}_{\{1\}}^{\mathbb{F}_{p^2}^*}(\overline{\mathbb{F}}_p) & \to & \mathrm{Ind}_{\tilde{U}}^{\tilde{B}}(\overline{\mathbb{F}}_p) \\ f & \mapsto & \left(\left(\begin{smallmatrix} a & b \\ 0 & e \end{smallmatrix}\right) \mapsto f(e) \right). \end{array}$$

The representation $\mathrm{Ind}_{\{1\}}^{\mathbb{F}_{p^2}^*}(\overline{\mathbb{F}}_p)$ is the regular representation of $\mathbb{F}_{p^2}^*$. This is a (p^2-1)-dimensional representation of an abelian group of order prime to p and hence it admits a decomposition into a direct sum of one-dimensional representations of $\mathbb{F}_{p^2}^*$. By a slight abuse of notation, the summands are the $\mathbb{F}_{p^2}^*$-modules $\overline{\mathbb{F}}_p^{\chi^d}$, where for $x \in \mathbb{F}_{p^2}^*, y \in \overline{\mathbb{F}}_p$, we have $x.y := x^d y$ with $0 \leq d \leq p^2-2$.

Proposition 2.3.4. *For all i, there is the following isomorphism of left $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-modules and left $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-modules respectively:*

1. $\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p) \cong \oplus_{d=0}^{p^2-2}\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\chi^d})$

2. $\mathrm{Ind}_{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p) \cong \oplus_{d=0}^{p^2-2}\mathrm{Ind}_{\tilde{B}\cap\tilde{S}}^{\tilde{S}}(\overline{\mathbb{F}}_p^{\chi^d})$.

Proof. Because of the bijection $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})\backslash\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \longleftrightarrow \tilde{U}\backslash\tilde{G}$ given by reducing modulo p, the transitivity of Ind, and the observation above, we have the following identifications of left $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-modules:

$$\begin{array}{rcl}\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p) & \cong & \mathrm{Ind}_{\tilde{U}}^{\tilde{G}}(\overline{\mathbb{F}}_p) \\ & \cong & \mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\mathrm{Ind}_{\tilde{U}}^{\tilde{B}}(\overline{\mathbb{F}}_p)) \\ & \cong & \mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\oplus_{d=0}^{p^2-2}(\overline{\mathbb{F}}_p^{\chi^d})) \\ & \cong & \oplus_{d=0}^{p^2-2}\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\chi^d}). \end{array}$$

For the second item, one uses the bijection $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn})\backslash\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \longleftrightarrow (\tilde{U}\cap\tilde{S})\backslash\tilde{S}$. □

We shall need a more explicit version of $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\chi^d})$. For $0 \leq d \leq p^2-2$, we define the following $\tilde{G}$-module which we denote by $\mathrm{U}_d(\overline{\mathbb{F}}_p)$:

$$\mathrm{U}_d(\overline{\mathbb{F}}_p) = \{f : \mathbb{F}_{p^2}^2 \to \overline{\mathbb{F}}_p : f(xa, xb) = x^d f(a,b)\, \forall\, x \in \mathbb{F}_{p^2}^*\}.$$

Next we define the following homomorphism

$$\begin{array}{rcl}\varphi : \mathrm{U}_d(\overline{\mathbb{F}}_p) & \to & \mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\chi^d}) \\ F & \mapsto & ((\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}) \mapsto F(c,e)).\end{array}$$

We shall show that it is an isomorphism of $\tilde{G}$-modules. It is well defined since

$$\varphi(F)((\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix})(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix})) = F(zc, ze) = z^d F(c,e) = \chi^d((\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}))\varphi(F)((\begin{smallmatrix} a & b \\ c & e \end{smallmatrix})).$$

It is also easy to see that φ is an $\tilde{G}$-homomorphism. In order to conclude that φ is an isomorphism, one can define the inverse ψ of φ as follows. We first note that for $c,\, e \in \mathbb{F}_{p^2}$ not both zero we can find $a, b \in \mathbb{F}_{p^2}$ such that $ae - bc \neq 0$. Hence an element $(c,e) \neq (0,0)$ gives rise to a matrix $(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix})$ in $\tilde{G}$. Another choice of a', b' with $a'e - b'c \neq 0$ amounts to multiply $(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix})$ from the left by a matrix of the form $(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix})$ which acts trivially on $\overline{\mathbb{F}}_p^{\chi^d}$. This implies that the map

$$\begin{array}{rcl}\psi : \mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\chi^d}) & \to & \mathrm{U}_d(\overline{\mathbb{F}}_p) \\ f & \mapsto & ((c,e) \mapsto f((\begin{smallmatrix} a & b \\ c & e \end{smallmatrix})),\end{array}$$

is well defined, that is, to mean that any choice of $a, b \in \mathbb{F}_{p^2}$ with $ae - bc \neq 0$ will do. Furthermore it is easy to verify that it is an $\tilde{G}$-homomorphism and it is the inverse of φ.

In the next remark, there is another proof of the isomorphism of $\tilde{G}$-modules : $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\chi^d}) \cong \mathrm{U}_d(\overline{\mathbb{F}}_p)$.

Remark 2.3.5. *We start with the identification of $\overline{\mathbb{F}}_p$-vector spaces*

$$\overline{\mathbb{F}}_p[X,Y]/(X^{p^2} - X, Y^{p^2} - Y) \cong \{f : \mathbb{F}_{p^2}^2 \to \overline{\mathbb{F}}_p\}$$

where $P(X,Y)$ maps to the function $(a,b) \mapsto P(a,b)$. To see this we observe that the spaces on both sides have dimensions p^4 as $\overline{\mathbb{F}}_p$-vector spaces. So, we just have to prove injectivity. To this end for any $x \in \mathbb{F}_{p^2}$ if the polynomial $f_x(Y) = P(x,Y) \in \overline{\mathbb{F}}_p[Y]$ vanishes for all $y \in \mathbb{F}_{p^2}$, then this means that Y and $Y^{p^2-1} - 1$ divide $f_x(Y)$ for all $x \in \mathbb{F}_{p^2}$. Because x is arbitrarily chosen we deduce that $Y^{p^2} - Y$ divides $P(X,Y)$. As the role of X and Y are symmetric, one obtains that $P(X,Y)$ lies in the ideal $(X^{p^2} - X, Y^{p^2} - Y)$. In fact this is an isomorphism of $\overline{\mathbb{F}}_p[\tilde{G}]$-modules. Let $\mathcal{W}(p, \overline{\mathbb{F}}_p) := \{f \in \overline{\mathbb{F}}_p[X,Y]/(X^{p^2} - X, Y^{p^2} - Y) : f((0,0)) = 0\}$. This module can be identified with $\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$ as $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-module, see [40] for details. Then $\mathrm{U}_d(\overline{\mathbb{F}}_p)$ is the subspace of homogeneous polynomial classes of degree

d in $\overline{\mathbb{F}}_p[X,Y]/(X^{p^2}-X, Y^{p^2}-Y)$ with $f((0,0))=0$. And as a graded $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-module $\mathcal{W}(p,\overline{\mathbb{F}}_p)$ decomposes as follows:

$$\mathcal{W}(p,\overline{\mathbb{F}}_p) = \oplus_{d=0}^{p^2-2}\mathrm{U}_d(\overline{\mathbb{F}}_p).$$

The isomorphism in Proposition 2.3.4 will permit us to obtain a better understanding of the non-semisimple $\tilde{G}$-module $\mathrm{U}_d(\overline{\mathbb{F}}_p)$. We shall turn to this among other things.

2.4 Irreducible $\tilde{G}$-modules

We keep the same notation as in the previous sections. We will prove here the main results. For an irreducible $\overline{\mathbb{F}}_p[\tilde{G}]$-module W, this is done by embedding a cohomology group with coefficients in W into another cohomology group with trivial coefficients roughly speaking.

First of all, we shall see how the irreducible $\overline{\mathbb{F}}_p[\tilde{G}]$-modules can be embedded in a twist of $\mathrm{U}_d(\overline{\mathbb{F}}_p)$. Let τ be the non-trivial automorphism of $\mathbb{F}_{p^2}$. For $0 \leq r,s \leq p-1, 0 \leq l,t, \leq p-1$, recall that when l and t are not both equal to $p-1$, the representations

$$V_{r,s}^{l,t}(\overline{\mathbb{F}}_p) := \mathrm{Sym}^r(\overline{\mathbb{F}}_p^2) \otimes_{\mathbb{F}_{p^2}} det^l \otimes_{\mathbb{F}_{p^2}} \mathrm{Sym}^s(\overline{\mathbb{F}}_p^2)^\tau \otimes_{\mathbb{F}_{p^2}} (det^t)^\tau,$$

exhaust all the irreducible $\overline{\mathbb{F}}_p[\tilde{G}]$-modules. Here, we identify $\mathrm{Sym}^r(\overline{\mathbb{F}}_p^2)$ with the homogeneous polynomials in the variables X,Y over $\overline{\mathbb{F}}_p$ of degree r which we denote by $\overline{\mathbb{F}}_p[X,Y]_r$. A matrix $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \tilde{G}$ acts from the left on $V_{r,s}^{l,t}(\overline{\mathbb{F}}_p)$ as follows: on the first factor $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right).X^iY^j := (aX+bY)^i(cX+dY)^j$ followed by multiplication by $(ad-bc)^l$ and on the second factor we apply first τ on $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ and proceed as for the first factor followed by multiplication by $(ad-bc)^{pt}$, e.g,

$$\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right).X^iY^j \otimes X^{i'}Y^{j'} := (ad-bc)^{l+pt}(aX+bY)^i(cX+dY)^j \otimes (a^pX+b^pY)^{i'}(c^pX+d^pY)^{j'}.$$

For $e \geq 0$, we write $\mathrm{U}_d^e(\overline{\mathbb{F}}_p)$ to mean the $\overline{\mathbb{F}}_p[\tilde{G}]$-module $\mathrm{U}_d(\overline{\mathbb{F}}_p)$ with the natural action of $\tilde{G}$ followed by multiplication by det^e, e.g, $\mathrm{U}_d^e(\overline{\mathbb{F}}_p) = \mathrm{U}_d(\overline{\mathbb{F}}_p) \otimes_{\mathbb{F}_{p^2}} det^e$.

Lemma 2.4.1. *We have the following embedding of left $\overline{\mathbb{F}}_p[\tilde{G}]$-modules*

$$\begin{aligned} \Psi : V_{r,s}^{q,t}(\overline{\mathbb{F}}_p) &\to \mathrm{U}_{r+ps}^{q+pt}(\overline{\mathbb{F}}_p) \\ f \otimes g &\mapsto ((a,b) \mapsto f(a,b)g(a^p,b^p)). \end{aligned}$$

Proof. In polynomial terms we can write $\Psi(f(X,Y) \otimes g(X,Y)) = f(X,Y)g(X^p,Y^p)$.

By definition of Ψ we have $\Psi(\sum f_i \otimes g_i) = \sum \Psi(f_i \otimes g_i)$. Now let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \tilde{G}$, we need to check that $\Psi(M.f \otimes M^\tau.g) = M.\Psi(f \otimes g)$. For $f(X,Y) = \sum_{n+m=r} a_{n,m} X^n Y^m$, $g(X,Y) = \sum_{l+k=s} b_{l,k} X^l Y^k$, then $M.f = (ad-bc)^q \sum_{n+m=r} a_{n,m}(aX+bY)^n(cX+dY)^m$ and $M^\tau.g = (ad-bc)^{pt} \sum_{l+k=s} b_{l,k}(a^pX+b^pY)^l(c^pX+d^pY)^k$. We set $\alpha = (ad-bc)^{q+pt}$. Hence by denoting $\Psi(M.(f \otimes g))$ as $(*)$, we have

$$\begin{aligned}
(*) &= \alpha \sum_{n+m=r} \sum_{l+k=s} a_{n,m}(aX+bY)^n(cX+dY)^m b_{l,k}(a^pX^p+b^pY^p)^l(c^pX^p+d^pY^p)^k \\
&= \alpha \sum_{n+m=r} \sum_{l+k=s} a_{n,m}(aX+bY)^n(cX+dY)^m b_{l,k}(aX+bY)^{pl}(cX+dY)^{pk} \\
&= M.\Psi(f \otimes g).
\end{aligned}$$

□

One would then like to have a more concrete description of the cokernel of Ψ. In other words, one has to compute the Jordan-Hölder series of $\mathrm{U}^e_{r+ps}(\overline{\mathbb{F}}_p)$.

Remark 2.4.2. *In the special case $s = 0$, the semi-simplification of $\mathrm{U}^e_r(k)$ for k a finite field can be obtained by immediate generalization of the case $k = \mathbb{F}_p$ which is treated for instance in [40]. But as it seems that this method does not apply when $s > 0$, we will naturally follow the Brauer character theory approach which gives the semisimplification of $\mathrm{U}^e_d(k)$ in complete generality.*

For our purpose we shall next see that

$$(\mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p))^{ss} = V^{0,0}_{r,s}(\overline{\mathbb{F}}_p) \oplus V^{r,s}_{p-r-1,p-1-s}(\overline{\mathbb{F}}_p) \oplus V^{0,s+1}_{r-1,p-2-s}(\overline{\mathbb{F}}_p) \oplus V^{r+1,0}_{p-r-2,s-1}(\overline{\mathbb{F}}_p).$$

From this we deduce the semisimplification of $\mathrm{U}^e_{r+ps}(\overline{\mathbb{F}}_p)$ by twisting.

The constituents of $\mathrm{U}^e_{r+ps}(\overline{\mathbb{F}}_p)$

Let k be a finite field and $\mathfrak{G} = \mathrm{GL}_2(k)$, and $\mathfrak{B}$ its Borel subgroup of upper triangular matrices. For a character ϕ of $\mathfrak{B}$ with values in $\overline{\mathbb{F}}_p$, we consider $\mathrm{Ind}^{\mathfrak{G}}_{\mathfrak{B}}(\overline{\mathbb{F}}_p^{\phi^d})$ where $0 \leq \ d \leq \sharp k - 2$. The semisimplication of $\mathrm{Ind}^{\mathfrak{G}}_{\mathfrak{B}}(\overline{\mathbb{F}}_p^{\phi^d})$ is computed in [16] via Brauer character theory. Given two homomorphisms χ_1, $\chi_2 : k^* \to \overline{\mathbb{Q}}^*$(or $\overline{\mathbb{Q}}_p^*$), one obtains a character of $\mathfrak{B}$ induced by χ_1, χ_2 as

$$\left(\begin{smallmatrix} a & b \\ 0 & e \end{smallmatrix}\right) \mapsto \chi_1(a)\chi_2(e).$$

Furthermore for $V = \overline{\mathbb{Q}}$ (or $\overline{\mathbb{Q}}_p$) let $\mathrm{I}(\chi_1, \chi_2) := \mathrm{Ind}_{\mathfrak{B}}^{\mathfrak{G}}(V^{\chi_1,\chi_2})$ where

$$\mathrm{Ind}_{\mathfrak{B}}^{\mathfrak{G}}(V^{\chi_1,\chi_2}) = \{f : \mathfrak{G} \to V : f(\left(\begin{smallmatrix} a & b \\ 0 & e \end{smallmatrix}\right) g) = \chi_1(a)\chi_2(e)f(g) \ \forall \left(\begin{smallmatrix} a & b \\ 0 & e \end{smallmatrix}\right) \in \mathfrak{B},\ g \in \mathfrak{G}\}.$$

This is a $(q+1)$-dimensional representation of $\tilde{G}$ where $q = \sharp k$. It is known as a principal series representation of $\mathfrak{G}$. Next let E be the set of embeddings $k \to \overline{\mathbb{F}}_p$. Then the complete list of irreducible $\overline{\mathbb{F}}_p$-representations of $\mathfrak{G}$ is given by:

$$R_{\overrightarrow{m},\overrightarrow{n}} = \otimes_{\tau\in E}(\mathrm{Sym}^{n_\tau - 1}(k^2)))^\tau \otimes (det^{m_\tau})^\tau \otimes \overline{\mathbb{F}}_p;$$

for integers $0 \leq m_\tau \leq p-1$ and $1 \leq n_\tau \leq p$ associated with each $\tau \in E$, and some n_τ is less than $p-1$. Here one makes the convention $\mathrm{Sym}^{-1}(k^2) = \{0\}$, the null module. Before we go further, note that in our notation we have

$$V_{r,s}^{l,t}(\overline{\mathbb{F}}_p) = R_{(l,t),(r+1,s+1)}$$

as irreducible $\mathfrak{G}$-modules.

To obtain the semisimplification of an $\overline{\mathbb{F}}_p$-representation of $\mathfrak{G}$, the approach is via Brauer character theory. One starts with a $\overline{\mathbb{Q}}_p$-representation W of $\mathfrak{G}$, and reduction modulo the maximal ideal of $\overline{\mathbb{Z}}_p$ yields an $\overline{\mathbb{F}}_p$-representation of $\mathfrak{G}$. More precisely for such a W, we know that there exists a $\overline{\mathbb{Z}}_p$-lattice L inside W invariant under the action of $\mathfrak{G}$. Then reduction of L modulo the maximal ideal of $\overline{\mathbb{Z}}_p$ gives rise to an $\overline{\mathbb{F}}_p$-representation whose Brauer character is the restriction of the character of W to the p-regular classes of $\mathfrak{G}$. In this way the semisimplification thus obtained is independent of the lattice L.

Any group homomorphism $\varphi : k^* \to \overline{\mathbb{Q}}_p^*$ can be written as $\varphi = \prod_\tau \tilde{\tau}^{a_\tau}$ with $0 \leq a_\tau \leq p-1$ and $\tilde{\tau}$ the Teichmüller lift of τ. Then the reduction of φ is $\bar{\varphi} = \prod_\tau \tau^{a_\tau}$. By a twist it suffices to consider the irreducible representation of the form $\mathrm{I}(1,\chi)$. Then it was shown in [16] that

Proposition 2.4.3 (Diamond). *Let $M = \mathrm{I}(1, \prod_\tau \tilde{\tau}^{a_\tau})$ with $0 \leq a_\tau \leq p-1$ for each $\tau \in E$. Let Frob be the Frobenius in E. Then the semisimplification of the reduction $\overline{M}$ of M is $\overline{M} \cong \oplus_{J\subset S} M_J$ with $M_J = R_{\overrightarrow{m}_J,\overrightarrow{n}_J}$, where*

$$m_{J,\tau} = \begin{cases} 0 & if\ \ \tau \in J \\ a_\tau + \delta_J(\tau) & otherwise; \end{cases} \qquad n_{J,\tau} = \begin{cases} a_\tau + \delta_J(\tau) & if\ \ \tau \in J \\ p - a_\tau - \delta_J(\tau) & otherwise; \end{cases}$$

with δ_J the characteristic function of $J^{(p)} = \{\tau \circ Frob : \tau \in J\}$.

We shall next specialize to our setting. We take $\mathfrak{G} = \tilde{G}$ and $\mathfrak{B} = \tilde{B}$. Let $\overline{\psi} : \mathbb{F}_{p^2} \hookrightarrow \overline{\mathbb{F}}_p$ be a fixed injection. We also denote by $\overline{\psi}$ the character obtained by restriction to $\mathbb{F}_{p^2}^*$. Now, let $\psi : \mathbb{F}_{p^2}^* \to \overline{\mathbb{Q}}_p^*$ be the Teichmüller lift of $\overline{\psi}$. The homomorphism ψ induces the character of $\tilde{B}$:

$$\tilde{B} \to \overline{\mathbb{Q}}_p^*;\ \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \mapsto \psi(z).$$

Via ψ, $\overline{\mathbb{Q}}_p$ is endowed with a structure of $\tilde{B}$-module which we denote $\overline{\mathbb{Q}}_p^{\psi}$: for $h \in \tilde{B}, x \in \overline{\mathbb{Q}}_p$ we have $h.x = \psi(h)x$. The $\overline{\mathbb{Q}}_p$-representation

$$\mathrm{I}(1,\psi) = \mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{Q}}_p^{\psi}) = \{f : \tilde{G} \to \overline{\mathbb{Q}}_p : f(hg) = \psi(h)f(g)\ \forall h \in \tilde{B},\ g \in \tilde{G}\}$$

of $\tilde{G}$ has reduction the $\overline{\mathbb{F}}_p$-representation $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\overline{\psi}})$ where by abuse of notation $\overline{\psi}$ is the character of $\tilde{B}$ induced by $\overline{\psi}$:

$$\tilde{B} \to \overline{\mathbb{F}}_p^*;\ \left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right) \mapsto \overline{\psi(z)}.$$

For $0 \leq d \leq p^2 - 2$, we consider the $\overline{\mathbb{F}}_p$-representations of $\tilde{G}$: $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\overline{\psi}^d})$. We write $d = r + ps$ with $0 \leq r, s \leq p-1$, $E = \{id, \tau\}$ so that $\overline{\psi}^d = id^r\tau^s$. From Proposition 2.4.3, we have

$$(\mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p))^{ss} = V_{r,s}^{0,0}(\overline{\mathbb{F}}_p) \oplus V_{p-r-1,p-1-s}^{r,s}(\overline{\mathbb{F}}_p) \oplus V_{r-1,p-2-s}^{0,s+1}(\overline{\mathbb{F}}_p) \oplus V_{p-r-2,s-1}^{r+1,0}(\overline{\mathbb{F}}_p);$$

where we have used the identification of $\mathrm{U}_d(\overline{\mathbb{F}}_p)$ with $\mathrm{Ind}_{\tilde{B}}^{\tilde{G}}(\overline{\mathbb{F}}_p^{\overline{\psi}^d})$ as $\overline{\mathbb{F}}_p[\tilde{G}]$-modules from page 30. We define the representation $W_{r,s}^{l,t}$ by the exact sequence

$$0 \to V_{r,s}^{l,t}(\overline{\mathbb{F}}_p) \to \mathrm{U}_{r+ps}^{l+pt}(\overline{\mathbb{F}}_p) \to W_{r,s}^{l,t} \to 0.$$

Thus the semisimplification of $W_{r,s}^{l,t}$ is

$$(W_{r,s}^{l,t})^{ss} = V_{p-r-1,p-s-1}^{r+l,s+t}(\overline{\mathbb{F}}_p) \oplus V_{r-1,p-s-2}^{l,s+1+t}(\overline{\mathbb{F}}_p) \oplus V_{p-r-2,s-1}^{r+l+1,t}(\overline{\mathbb{F}}_p).$$

2.4.1 Some invariants

Let $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ be the congruence subgroups of $G(F)$ defined in Section 2.2. We view $\overline{\mathbb{F}}_p$ as a trivial left $\tilde{G}$-module. We need to remind us once more how Hecke operators act on the degree zero group cohomology. Let $g \in \Delta_1^{\mathfrak{q}}(\mathfrak{n})$ where $\Delta_1^{\mathfrak{q}}(\mathfrak{n})$ is the subset of $Mat_2(\hat{\mathcal{O}})_{\neq 0}$ defined in Section 2.2. From Lemma 2.2.3, we have that for each $i, 1 \leq i \leq h$, there are a unique index j_i and a matrix $\beta_i \in \Lambda_{1,[\mathfrak{b}_i]}^{\mathfrak{c}}(\mathfrak{n})$ such that $K_1(\mathfrak{n})gK_1(\mathfrak{n}) = K_1(\mathfrak{n})g_{j_i}\beta_i g_i^{-1}K_1(\mathfrak{n})$

with g_i the matrices corresponding to the ideal classes as defined in Subsection 2.2.2.

Let M be a finite dimensional left $\overline{\mathbb{F}}_p[\tilde{G}]$-module. We have seen that the Hecke operator corresponding to the double coset $K_1(\mathfrak{n})gK_1(\mathfrak{n})$ which we have denoted as T_g sends $(m_1,\cdots,m_h)\in\oplus_{i=1}^h \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}),M)$ to $(n_1,\cdots,n_h)$ with $n_{j_i}=T_{\beta_i}.m_i$. Here T_{β_i} is the Hecke operator corresponding to the double coset $\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. Explicitly one defines $\Gamma''^{,\beta_i}_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})=\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})\beta_i^{-1}\cap\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})$, then we have

$$\begin{aligned} T_{\beta_i}:\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}),M) &\to \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}),M)\\ m &\mapsto \sum_{\lambda\in\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})/\Gamma''^{,\beta_i}_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})} (\lambda\beta_i).m. \end{aligned}$$

This being said here are the $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ and $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-invariants for $\mathrm{U}^e_d(\overline{\mathbb{F}}_p)$, $V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)$, $(W^{l,t}_{r,s})^{ss}$ and $W^{l,t}_{r,s}$.

Lemma 2.4.4. *Let d and n be integers greater than or equal to zero. Then one has*

1. *for all $n\geq 0$, one has*

$$\oplus_{i=1}^h \mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}),\mathrm{U}^n_d(\overline{\mathbb{F}}_p))=\begin{cases}\oplus_{i=1}^h\overline{\mathbb{F}}_p & \text{if } d\equiv 0 \pmod{p^2-1}\\ 0 & \text{otherwise}\end{cases}$$

   as $\overline{\mathbb{F}}_p$-vector spaces.

2. $\oplus_{i=1}^h \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}),\mathrm{U}^n_d(\overline{\mathbb{F}}_p))=\begin{cases}\oplus_{i=1}^h\overline{\mathbb{F}}_p & \text{if } d\equiv 0 \pmod{p^2-1} \text{ and } (\mathcal{O}^*)^n=1\\ 0 & \text{otherwise}\end{cases}$

   as $\overline{\mathbb{F}}_p$-vector spaces.

3. *the Hecke operator T_g acts on $(m_1,\cdots,m_h)\in\oplus_{i=1}^h\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}),\mathrm{U}^n_d(\overline{\mathbb{F}}_p))=\oplus_{i=1}^h\overline{\mathbb{F}}_p$, where $d\equiv 0 \pmod{p^2-1}$ and $(\mathcal{O}^*)^n=1$, by sending m_i to n_{j_i} with*

$$n_{j_i}=[\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}):\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})\beta_i^{-1}\cap\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})]m_i.$$

Proof. As for the first item, because $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ reduces modulo $\mathfrak{p}$ to $\tilde{S}$ and since we can identify $\mathrm{U}^n_d(\overline{\mathbb{F}}_p)$ with $\mathrm{U}_d(\overline{\mathbb{F}}_p)$ as $\tilde{S}$-module, we see that the invariants do not depend on the values of n. Having this, it is suitable to view $\mathrm{U}_d(\overline{\mathbb{F}}_p)$ as the set of $\overline{\mathbb{F}}_p$-valued functions on $\mathbb{F}^2_{p^2}$ and homogeneous of degree d. Observe first that a non-null constant function belongs to $(\mathrm{U}_d(\overline{\mathbb{F}}_p))^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}$ if and only if $d\equiv 0 \pmod{p^2-1}$. Any nonzero $f\in(\mathrm{U}_d(\overline{\mathbb{F}}_p))^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}$ is a constant function. Indeed let $(0,0)\neq(a,b),(a',b')\in\mathbb{F}^2_{p^2}$ and suppose that $f(a,b) = x \neq f(a',b') = y$. Now since (a,b), $(a',b') \neq (0,0)$ there are $c, e, c', d' \in \mathbb{F}_{p^2}$ such that $\left(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \tilde{S}$. Then $(a',b') = (a,b)\left(\begin{smallmatrix} e & -b \\ -c & a \end{smallmatrix}\right)\left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$. Therefore $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ acts transitively on $\mathbb{F}^2_{p^2} - \{(0,0)\}$. Indeed from Lemma 2.3.1, reduction modulo $\mathfrak{p}$ is a surjective homomorphism $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \twoheadrightarrow \tilde{S}$. So we have $y = f(a',b') = f((a,b)\gamma) = \gamma f((a,b)) = f(a,b) = x$, contradicting the hypothesis $x \neq y$. Hence $f \in (\mathrm{U}_d(\overline{\mathbb{F}}_p))^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}$ if and only if f is constant.

For the second item one firstly observes that a non-null constant function belongs to $(\mathrm{U}^n_d(\overline{\mathbb{F}}_p))^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}$ if and only if $d \equiv 0 \pmod{p^2-1}$ and $(\mathcal{O}^*)^n = 1$. From here the same proof as the one given for the first item applies.

For the third item, let $f_x \in (\mathrm{U}^n_d(\overline{\mathbb{F}}_p))^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}$ with $f(a,b) = x$ for all $(a,b) \in \mathbb{F}^2_{p^2} - \{(0,0)\}$ and let given $\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = \amalg_k \delta_k \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. Then

$$T_{\beta_i}.f_x(a,b) = \sum \delta_k\beta_i.f_x(a,b) = [\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}) : \beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})\beta_i^{-1} \cap \Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})]x.$$

From this what we have claimed follows. □

We also have the following

Lemma 2.4.5. *Let $0 \leq r, s \leq p-1$, and let l, t be integers greater than or equal to zero. Then one has*

1. $$\oplus^h_{1=i}\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) = \begin{cases} \oplus^h_{1=i}\overline{\mathbb{F}}_p & \text{if } r = s = 0 \text{ and for all } l,t \\ 0 & \text{otherwise} \end{cases}$$
   as $\overline{\mathbb{F}}_p$-vector spaces.

2. $$\oplus^h_{1=i}\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) = \begin{cases} \oplus^h_{1=i}\overline{\mathbb{F}}_p & \text{if } r = s = 0 \text{ and } (\mathcal{O}^*)^{l+pt} = 1 \\ 0 & \text{otherwise} \end{cases}$$
   as $\overline{\mathbb{F}}_p$-vector spaces.

3. *the Hecke operator T_g acts on $(m_1, \cdots, m_h)$ from $\oplus^h_{i=1}\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{0,0}(\overline{\mathbb{F}}_p))$ which is equal to $\oplus^h_{i=1}\overline{\mathbb{F}}_p$ when $(\mathcal{O}^*)^{l+pt} = 1$, by sending m_i to n_{j_i} with $n_{j_i} = [\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}) : \beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})\beta_i^{-1} \cap \Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})]m_i$.*

Proof. Firstly when $r = s = 0$, and for all l, t then $V^{l,t}_{r,s}(\overline{\mathbb{F}}_p) = \overline{\mathbb{F}}_p$ as $\tilde{S}$-module. By definition we have $\overline{\mathbb{F}}_p^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})} = \overline{\mathbb{F}}_p$. Otherwise use the fact that $V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)$ is irreducible as $\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-module.

The second item is proved similarly. The statement about the Hecke action is verified similarly as in the proof of Lemma 2.4.4. □

Lemma 2.4.6. *Let $0 \le r, s \le p-1, e := l + tp, e_1 := e + p(p-1), e_2 := e + p - 1 \ge 0$ and let f be the order of $\mathcal{O}^*$. The following isomorphisms of $\overline{\mathbb{F}}_p$-vectors spaces hold:*

1. $$\oplus_{1=i}^h \mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), (W_{r,s})^{ss}) = \begin{cases} \oplus_{1=i}^h \overline{\mathbb{F}}_p & \text{if} \begin{cases} r = s = p-1 \text{ or} \\ r = 1, s = p-2 \text{ or} \\ r = p-2, s = 1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

2. *suppose that $(r \neq 1$ or $s \neq p-2)$ and $(r \neq p-2$ or $s \neq 1)$; then we have*
$$\oplus_{1=i}^h \mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W_{r,s}) = \begin{cases} \oplus_{1=i}^h \overline{\mathbb{F}}_p & \text{if } r = s = p-1 \\ 0 & \text{otherwise} \end{cases}$$

3. $$\oplus_{1=i}^h \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), (W^{l,t}_{r,s})^{ss}) = \begin{cases} \oplus_{1=i}^h \overline{\mathbb{F}}_p & \text{if} \begin{cases} r = s = p-1 \text{ and } f \mid e \text{ or} \\ r = 1, s = p-2 \text{ and } f \mid e_1 \text{ or} \\ r = p-2, s = 1 \text{ and } f \mid e_2 \end{cases} \\ 0 & \text{otherwise.} \end{cases}$$

4. *suppose that $(r \neq 1$ or $s \neq p-2$ or $f \nmid e_1)$ and $(r \neq p-2$ or $s \neq 1$ or $f \nmid e_2)$; then we have*
$$\oplus_{1=i}^h \mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W^{l,t}_{r,s}) = \begin{cases} \oplus_{1=i}^h \overline{\mathbb{F}}_p & \text{if } r = s = p-1 \text{ and } f \mid e \\ 0 & \text{otherwise} \end{cases}$$

Lastly, the Hecke action on these spaces is as in the previous lemmas.

Proof. We have $(W_{r,s})^{\mathrm{ss}} = V^{r,s}_{p-r-1,p-s-1}(\overline{\mathbb{F}}_p) \oplus V^{0,s+1}_{r-1,p-s-2}(\overline{\mathbb{F}}_p) \oplus V^{r+1,0}_{p-r-2,s-1}(\overline{\mathbb{F}}_p)$. From the above lemma we know that $\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{r,s}_{p-r-1,p-s-1}(\overline{\mathbb{F}}_p))$ is non zero only when $r = s = p-1$. Indeed $V^{r,s}_{p-1-r,p-s-1}(\overline{\mathbb{F}}_p) \cong \overline{\mathbb{F}}_p$ as $\tilde{S}$-modules if and only if $r = p-1, s = p-1$. In this case we have $(W_{r,s})^{\mathrm{ss}} = V^{r,s}_{p-r-1,p-s-1}(\overline{\mathbb{F}}_p) \cong \overline{\mathbb{F}}_p$. Therefore we obtain that

$$\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), (W_{r,s})^{\mathrm{ss}}) = \overline{\mathbb{F}}_p.$$

From the same lemma $V^{0,s+1}_{r-1,p-s-2}(\overline{\mathbb{F}}_p)$ has non zero invariants only when $r = 1, s = p-2$. In this case we have

$$(W_{1,p-2})^{\mathrm{ss}} = V^{1,p-2}_{p-2,1}(\overline{\mathbb{F}}_p) \oplus V^{0,p-1}_{0,0}(\overline{\mathbb{F}}_p) \oplus V^{2,0}_{p-3,p-3}(\overline{\mathbb{F}}_p).$$

From this one has
$$\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}),(W_{1,p-2})^{\mathrm{ss}})=\overline{\mathbb{F}}_p.$$
From the same lemma the invariants of $V^{r+1,0}_{p-r-2,s-1}(\overline{\mathbb{F}}_p)$ are non zero if and only if $r=p-2, s=1$. Similarly we obtain that

$$\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}),(W_{p-2,1})^{\mathrm{ss}})=\overline{\mathbb{F}}_p.$$

As for the second item, when $r=s=p-1$, then $(W_{r,s})^{\mathrm{ss}}=W_{r,s}$. Otherwise, from $\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}),(W_{r,s})^{ss})=0$, so does $\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}),W_{r,s})$.
The remaining items are proved in a similar fashion. □

In the cases ($r=1, s=p-2$ and $f\mid e_1$) or ($r=p-2, s=1$ and $f\mid e_2$), further analysis is needed.

We shall next discuss the case $r=1, s=p-2$ and $f\mid e_1$ in detail as it is symmetric to the remaining one. We suppose in addition that $p>5$. So the representation $V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p)$ has dimension $2(p-1)$ and we identify it with its image in $\mathrm{U}^{l+pt}_{(p-1)^2}(\overline{\mathbb{F}}_p)$. Inside $\mathrm{U}^{l+pt}_{(p-1)^2}(\overline{\mathbb{F}}_p)$ lies the submodule M generated by the homogeneous monomials of degree $(p-1)^2$. The dimension of M is $(p-1)^2+1$ and it contains $V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p)$ as submodule. By dimensional consideration (it is here that we need to have $p>5$ to avoid discussing many cases), one deduces an exact sequence of $\overline{\mathbb{F}}_p[\tilde{G}]$-modules

$$0\to V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p)\to M\to V^{2+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p)\to 0.$$

Indeed from

$$(W^{l,t}_{1,p-2})^{\mathrm{ss}}=V^{l,p-1+t}_{0,0}(\overline{\mathbb{F}}_p)\oplus V^{2+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p)\oplus V^{1+l,p-2+t}_{p-2,1}(\overline{\mathbb{F}}_p),$$

we know that the constituents of any submodule of $W^{l,t}_{1,p-2}$ are among the representations $V^{l,p-1}_{0,0}(\overline{\mathbb{F}}_p), V^{2+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p)$ and $V^{1+l,p-2+t}_{p-2,1}(\overline{\mathbb{F}}_p)$. From the equality $(p-1)^2+1=(p-2)^2+2(p-1)$, it follows that
$$M/V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p)\cong V^{2+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p).$$
Therefore $V^{2+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p)$ is a submodule of $\mathrm{U}^{l+pt}_{(p-1)^2}/V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p)=(W^{l,t}_{1,p-2})^{\mathrm{ss}}$.
Next we can realize the module $V^{l,p-1+t}_{0,0}(\overline{\mathbb{F}}_p)$ as submodule of $(W^{l,t}_{1,p-2})^{\mathrm{ss}}$ by sending 1 to

the class $X^{p(p-1)}Y^{p(p-1)} + V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p)$. To see this we define

$$\begin{array}{rcl}\varphi : V^{l,p-1+t}_{0,0}(\overline{\mathbb{F}}_p) & \to & \mathrm{U}^{l+pt}_{(p-1)^2}(\overline{\mathbb{F}}_p)/V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p) \\ 1 & \mapsto & X^{p(p-1)}Y^{p(p-1)} + V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p).\end{array}$$

Then for $g = \left(\begin{smallmatrix} a & b \\ c & e \end{smallmatrix}\right) \in \tilde{G}$, we need to check that $\varphi(1) = g.\varphi(1)$. We have

$$g.X^{p(p-1)}Y^{p(p-1)} = (ae-bc)^{l+pt}(a^pX^p + b^pY^p)^{p-1}(c^pX^p + e^pY^p)^{p-1}.$$

The latter polynomial is a linear combination of the monomials $X^{2p^2-2p-i}Y^i$ with $p|i$. For all multiples i of p less or equal to $2p(p-1)$ except $p(p-1)$ the monomials $X^{2p^2-2p-i}Y^i$ belong to $V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p)$. Indeed let $i = pk$, we recall the relations $X^{p^2} = X, Y^{p^2} = Y$ in $\mathrm{U}^{l+pt}_{(p-1)^2}(\overline{\mathbb{F}}_p)$, then we have

$$X^{2p^2-2p-pk}Y^{pk} = X^{p^2-2p-pk}X^{p^2}Y^{pk} = X^{(p-1)^2-pk}Y^{pk} \in V^{l,t}_{1,p-2}(\overline{\mathbb{F}}_p).$$

Therefore $g.\varphi(1) \equiv \varphi(1) \pmod{V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)}$. This implies that the direct sum $V^{l,p-1}_{0,0}(\overline{\mathbb{F}}_p) \oplus V^{1+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p)$ is a submodule of $W^{l,t}_{1,p-2}$. Thus we get an exact sequence

$$0 \to V^{l,p-1+t}_{0,0}(\overline{\mathbb{F}}_p) \oplus V^{2+l,t}_{p-3,p-3}(\overline{\mathbb{F}}_p) \to (W^{l,t}_{1,p-2}) \to V^{1+l,p-2+t}_{p-2,1}(\overline{\mathbb{F}}_p) \to 0.$$

Hence for ($r = 1, s = p-2$ and $l + pt \equiv p-1 \pmod{p^2-1}$), we obtain that

$$\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W^{l,t}_{r,s}) = \overline{\mathbb{F}}_p.$$

For $(r = p-2, s = 1)$ and $l + pt \equiv 1 - p \pmod{p^2-1}$, similar arguments yield

$$\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W^{l,t}_{r,s}) = \overline{\mathbb{F}}_p.$$

In summary we have the following

Lemma 2.4.7. *Let $p > 5$ and $e := l + pt$. Then we have*

1. $\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W_{r,s}) = \overline{\mathbb{F}}_p$ *if* $\begin{cases} r = 1, s = p-2 \text{ or} \\ r = p-2, s = 1 \end{cases}$

2. $\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W^{l,t}_{r,s}) = \overline{\mathbb{F}}_p$ *if* $\begin{cases} r = 1, s = p-2 \text{ and } f \mid p(p-1) + e \text{ or} \\ r = p-2, s = 1 \text{ and } f \mid e + p - 1. \end{cases}$

Some indexes

Let $\mathfrak{t}$ be a finite place coprime with $\mathfrak{pn}$. The matrix $g \in Mat_2(\hat{\mathcal{O}})$ which has at the $\mathfrak{t}$-th place the matrix $\left(\begin{smallmatrix} \pi_\mathfrak{t} & 0 \\ 0 & 1 \end{smallmatrix}\right)$ where $\pi_\mathfrak{t}$ is a uniformizer of $\mathcal{O}_\mathfrak{t}$ and in all the remaining places has the identity matrix, belongs to $\Delta_1^\mathfrak{t}(\mathfrak{n})$. We shall fix in this sub-section such a g. Because

$$\begin{pmatrix} \pi_\mathfrak{t}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \left(\begin{smallmatrix} \pi_\mathfrak{t} & 0 \\ 0 & 1 \end{smallmatrix}\right) = \begin{pmatrix} a & \pi_\mathfrak{t}^{-1} b \\ \pi_\mathfrak{t} c & d \end{pmatrix};$$

we deduce that

$$K'_{1,g^{-1}}(\mathfrak{n}) := g^{-1} K_1(\mathfrak{n}) g \cap K_1(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in K_1(\mathfrak{n}) : \pi_\mathfrak{t} \mid c_\mathfrak{t} \right\}.$$

Consider also the subgroup

$$K_1^1(\mathfrak{n}) = \{\alpha \in K_1(\mathfrak{n}) : det(\alpha) = 1\}.$$

Similarly as in Lemma 2.3.1, one can prove that reduction modulo $\mathfrak{t}$ provides us with a surjective homomorphism

$$K_1^1(\mathfrak{n}) \twoheadrightarrow \mathrm{SL}_2(\mathcal{O}/\mathfrak{t}).$$

From this we deduce that we have a surjective map

$$\begin{array}{rcl} K_1(\mathfrak{n}) & \twoheadrightarrow & \mathbb{P}^1(\mathcal{O}/\mathfrak{t}) \\ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) & \mapsto & (c : d). \end{array}$$

This map is surjective because there is a surjective map $\mathrm{SL}_2(\mathcal{O}/\mathfrak{t}) \twoheadrightarrow \mathbb{P}^1(\mathcal{O}/\mathfrak{t})$; $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto (c : d)$ and also a surjective map $K_1(\mathfrak{n}) \supset K_1^1(\mathfrak{n}) \twoheadrightarrow \mathrm{SL}_2(\mathcal{O}/\mathfrak{t})$. Since the subgroup $K'_{1,g^{-1}}(\mathfrak{n})$ is the subset of all elements that are mapped to $(0 : 1)$, we deduce that we have a bijection

$$K'_{1,g^{-1}}(\mathfrak{n}) \backslash K_1(\mathfrak{n}) \longleftrightarrow \mathbb{P}^1(\mathcal{O}/\mathfrak{t}).$$

Therefore we obtain the index $[K_1(\mathfrak{n}) : K'_{1,g^{-1}}(\mathfrak{n})] = N(\mathfrak{t}) + 1$. Recall the definition $\Gamma'^{,\beta_i^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n}) := \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \cap \beta_i^{-1} \Gamma_{1,[\mathfrak{t}^{-1}\mathfrak{b}_i]}(\mathfrak{n}) \beta_i$. Similarly as $Y_{K_1(\mathfrak{n})}$ decomposes into disjoint union of its connected component $\amalg_{i=1}^h \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \backslash \mathbb{H}_3$, $Y_{K'_{1,g^{-1}}(\mathfrak{n})}$ decomposes as follows. We have $Y_{K'_{1,g^{-1}}(\mathfrak{n})} = \amalg_{i=1}^h \Gamma'^{,\beta_i^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \backslash \mathbb{H}_3$. Indeed we know that the connected components of $Y_{K'_{1,g^{-1}}(\mathfrak{n})}$ are $\Gamma''_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \backslash \mathbb{H}_3$ where $\Gamma''_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = g_i K'_{1,g^{-1}}(\mathfrak{n}) g_i^{-1} \cap G(F) = g_i g^{-1} K_1(\mathfrak{n}) g g_i^{-1} \cap \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. Recall that $\beta_i = g_{j_i} g g_i^{-1} k_i = \left(\begin{smallmatrix} y_i & 0 \\ 0 & 1 \end{smallmatrix}\right) \in G(F)$ with $k_i = \left(\begin{smallmatrix} u_i & 0 \\ 0 & 1 \end{smallmatrix}\right) \in g_i K_1(\mathfrak{n}) g_i^{-1}$.

For $\sigma \in \Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}) = g_{j_i}K_1(\mathfrak{n})g_{j_i}^{-1} \cap G(F)$ we have

$$\begin{aligned} \beta_i^{-1}\sigma\beta_i &\in \beta_i^{-1}g_{j_i}K_1(\mathfrak{n})g_{j_i}^{-1}\beta_i \cap G(F) \\ &= k_i^{-1}g_ig^{-1}g_{j_i}^{-1}g_{j_i}K_1(\mathfrak{n})g_{j_i}g_{j_i}^{-1}gg_i^{-1}k_i \cap G(F) \\ &= g_ig^{-1}K_1(\mathfrak{n})gg_i^{-1} \cap G(F) \end{aligned}$$

where we have used the facts that g_i, g, k_i commute and $k_i, k_i^{-1} \in g_iK_1(\mathfrak{n})g_i^{-1}$. This means that $\beta_i^{-1}\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})\beta_i = \Gamma''_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. Therefore we deduce that

$$\Gamma'^{,\beta_i^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \cap \beta_i^{-1}\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})\beta_i = \Gamma''_{1,[\mathfrak{b}_i]}(\mathfrak{n}).$$

So we have the following projection map

$$\begin{array}{c} Y_{K'_{1,g^{-1}}(\mathfrak{n})} = \amalg_{i=1}^{h}\Gamma'^{,\beta_i^{-1}}_{1,[\mathfrak{b}_i]}(\mathfrak{n})\backslash\mathbb{H}_3 \\ \big\downarrow s_g \\ Y_{K_1(\mathfrak{n})} = \amalg_{i=1}^{h}\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})\backslash\mathbb{H}_3. \end{array}$$

The map s_g is of degree $[K_1(\mathfrak{n}) : K'_{1,g^{-1}}(\mathfrak{n})] = N(\mathfrak{t}) + 1$. The maps induced by s_g on the connected components are also of degree $N(\mathfrak{t}) + 1$. The discussion we just made implies that for β_i corresponding to g, that is to mean $K_1(\mathfrak{n})gK_1(\mathfrak{n}) = K_1(\mathfrak{n})g_{j_i}^{-1}\beta_ig_iK_1(\mathfrak{n})$ where j_i is the unique index such that the ideal $(det(g_{j_i}gg_i^{-1}))$ is principal, the following holds.

Lemma 2.4.8. *Keeping the same assumptions as above, then for any ideal* $\mathfrak{n}$ *coprime with* $\mathfrak{t} = (det(g))$*, we have*

$$[\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n}) : \beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})\beta_i^{-1} \cap \Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})] = N(\mathfrak{t}) + 1.$$

Therefore, the Hecke eigenvalue corresponding to the action of $T_\mathfrak{t}$ on the $\overline{\mathbb{F}}_p$-vector space $\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p))$ where $\mathfrak{t}$ is a prime ideal coprime with $\mathfrak{pn}$ is $N(\mathfrak{t}) + 1$. Hence eigenvalue systems coming from the $\overline{\mathbb{F}}_p$-vector space $\oplus_{i=1}^{h}\mathrm{H}^0(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p))$ are Eisenstein because the semisimplification of the attached Galois representations is the direct sum of the cyclotomic character and the trivial character.

As we shall make use of Shapiro's isomorphism, we need to verify that it is compatible with the Hecke action on group cohomology. From the above discussion, we deduce that we can choose identical coset representatives for the double cosets

$$\Gamma_{1,[\mathfrak{t}^{-1}\mathfrak{b}_i]}(\mathfrak{pn})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})/\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn}) \text{ and for } \Gamma_{1,[\mathfrak{t}^{-1}\mathfrak{b}_i]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}).$$

This will be used for the compatibility of the Hecke action with the Shapiro's isomorphism.

Compatibility of Shapiro's lemma with the Hecke action

Recall that when $\Gamma' < \Gamma$ are congruence subgroups and M is a Γ'-module then Shapiro's isomorphism reads as

$$\mathrm{H}^*(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma}(M)) \cong \mathrm{H}^*(\Gamma', M).$$

It is the isomorphism induced by the restriction $j : \Gamma' \hookrightarrow \Gamma$ and the homomorphism

$$\begin{array}{rcl} \phi : \mathrm{Ind}_{\Gamma'}^{\Gamma}(M) & \to & M \\ f & \mapsto & f(1). \end{array}$$

Therefore in terms of cocycles we have

$$\begin{array}{rcl} Sh : \mathrm{H}^*(\Gamma, \mathrm{Ind}_{\Gamma'}^{\Gamma}(M)) & \to & \mathrm{H}^*(\Gamma', M) \\ c & \mapsto & \phi \circ c \circ j. \end{array}$$

From Subsection 2.2.2, we know that for $g \in \Delta_1^{\mathfrak{a}}(\mathfrak{pn})$ with $\mathfrak{a}$ coprime with $\mathfrak{pn}$, any set of orbit representatives of the orbit space $K_1(\mathfrak{pn})gK_1(\mathfrak{pn})/K_1(\mathfrak{pn})$ belongs to $\Delta_1^{\mathfrak{a}}(\mathfrak{pn})$ where $\mathfrak{a} = (det(g))\mathcal{O}$. In a similar fashion any set of orbit representatives of $K_1(\mathfrak{n})gK_1(\mathfrak{n})/K_1(\mathfrak{n})$ belongs to $\Delta_1^{\mathfrak{a}}(\mathfrak{n})$. We also obtain that any set of representatives of the orbit space $\Gamma_{1,[\mathfrak{b}_{j_i}]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ belong to $\Lambda_{1,[\mathfrak{b}_i]}^{\mathfrak{c}}(\mathfrak{n})$ when β_i is from $\Lambda_{1,[\mathfrak{b}_i]}^{\mathfrak{c}}(\mathfrak{n})$. For the forthcoming statement we need to recall some important facts. On page 29, we saw that reduction modulo $\mathfrak{p}$ provides us the following isomorphism of $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-modules:

$$\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p) \cong \mathrm{Ind}_{\tilde{U}}^{\tilde{G}}(\overline{\mathbb{F}}_p) \cong \mathrm{Ind}_{\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$$

where $\tilde{U} = \{\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in \tilde{G}\} \subset \tilde{B}$ with $\tilde{B}$ the Borel subgroup of $\tilde{G}$ and $\overline{\mathbb{F}}_p$ is endowed with the structure of a trivial left $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-module. The left action of the latter on $\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$ is as follows: for $\gamma \in \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ and $f \in \mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$ we have $(\gamma.f)(h) := f(h\gamma)$. By definition

$$\mathrm{Ind}_{\tilde{U}}^{\tilde{G}}(\overline{\mathbb{F}}_p) := \{f : \tilde{G} \to \overline{\mathbb{F}}_p : f(uh) = uf(h) = f(h), \forall\, u \in \tilde{U}, h \in \tilde{G}\},$$

that is the collection of all the $\tilde{U}$-left invariant maps from $\tilde{G}$ to $\overline{\mathbb{F}}_p$. Because for each $i = 1, \cdots, h$, any element $\lambda \in \Lambda_{1,[\mathfrak{b}_i]}^{\mathfrak{c}}(\mathfrak{pn})$ has its reduction belonging to $\tilde{U}$, we derive that any $f \in \mathrm{Ind}_{\tilde{U}}^{\tilde{G}}(\overline{\mathbb{F}}_p)$ satisfies

$$f(\lambda) = f(1).$$

Proposition 2.4.9. *Let $\mathfrak{a}$ be a prime ideal coprime with $\mathfrak{p}\mathfrak{n}$. Let $T_g = T_{\mathfrak{a}}$ be the Hecke operator associated with $g \in \Delta_1^{\mathfrak{a}}(\mathfrak{n})$ where $\mathfrak{a} = det(g)\mathcal{O}$ the ideal corresponding to g. Explicitly g is the matrix with the identity matrix in all finite places expect at the $\mathfrak{a}$-place where there is the matrix $\left(\begin{smallmatrix} \pi_{\mathfrak{a}} & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Here $\pi_{\mathfrak{a}}$ is a uniformizer of $\mathcal{O}_{\mathfrak{a}}$. Then the following diagram*

$$\begin{array}{ccc}
\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)) & \xrightarrow{Sh} & \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p) \\
\Big\downarrow T_{\beta_i} & & \Big\downarrow T_{\beta_i} \\
\mathrm{H}^1(\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{Ind}_{\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})}^{\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)) & \xrightarrow{Sh} & \mathrm{H}^1(\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p)
\end{array}$$

is well defined and is commutative.

Proof. The arrow that could potentially not be well defined is the left vertical arrow. However this is not an issue since for each i reduction modulo p yields that the coefficients are isomorphic $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$-modules as we just recalled above. We next verify the commutativity of the diagram. We set $M := \overline{\mathbb{F}}_p$. Let β_i corresponding to g as provides by Lemma 2.2.3. From a decomposition of the double coset $\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}) = \amalg_r \delta_r \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})$, we know that

$$\begin{aligned}
T_{\beta_i} : \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), M) \quad &\to \quad \mathrm{H}^1(\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), M) \\
c \mapsto \quad &\left(w \mapsto \sum_r \delta_r.c(\delta_r^{-1} w \delta_{s_r})\right)
\end{aligned}$$

where s_r is the unique index such that $\delta_r^{-1} w \delta_{s_r} \in \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})$. Therefore for c a cocycle from $\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(M))$ and w from $\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})$ we have

$$\big(T_{\beta_i} \circ Sh(c)\big)(w) = \sum_r \delta_r \big(c(\delta_r^{-1} w \delta_{s_r})(1)\big).$$

Let $\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = \amalg_s \lambda_s \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ so that

$$\begin{aligned}
T_{\beta_i} : \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(M)) \quad &\to \quad \mathrm{H}^1(\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{Ind}_{\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})}^{\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n})}(M)) \\
c \quad &\mapsto \quad \left(w \mapsto \sum_s \lambda_s c(\lambda_s^{-1} w \lambda_{n_s})\right).
\end{aligned}$$

Here n_s is the unique index such that $\lambda_s^{-1} w \lambda_{n_s} \in \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ for $w \in \Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n})$. We also have

$$\begin{array}{rcl} Sh : \mathrm{H}^1(\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{Ind}_{\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n})}(M)) & \to & \mathrm{H}^1(\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{pn}), M) \\ c & \mapsto & (w \mapsto c(w)(1)). \end{array}$$

Now a set of coset representatives of $\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{pn})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})/\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})$ can be chosen to be identical to coset representatives of $\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. So choose $\delta_r \in \Lambda^{\mathfrak{c}}_{1,[\mathfrak{b}_i]}(\mathfrak{pn})$ such that $\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{pn})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn}) = \amalg_r^k \delta_r \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})$. Also take $\lambda_1 = \delta_1, \cdots, \lambda_k = \delta_k$ all belonging to $\Lambda^{\mathfrak{c}}_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ such that $\Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{n})\beta_i\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}) = \amalg_r^k \lambda_r \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})$. Let c be a cocycle from $\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(M))$ and $w \in \Gamma_{1,[\mathfrak{a}^{-1}\mathfrak{b}_i]}(\mathfrak{pn})$. We have then

$$\big(Sh \circ T_{\beta_i}(c)\big)(w) = \sum_r^k \big(\lambda_r c(\lambda_r^{-1} w \lambda_{n_r})\big)(1).$$

Because for each $i = 1, \cdots, h$, the action of $\Lambda^{\mathfrak{c}}_{1,[\mathfrak{b}_i]}(\mathfrak{n})$ on $\overline{\mathbb{F}}_p$ is trivial, we obtain that

$$\delta_r\big(c(\delta_r^{-1} w \delta_{n_r})(1)\big) = c(\delta_r^{-1} w \delta_{n_r})(1).$$

But also since λ_r reduces modulo $\mathfrak{pn}$ to an element in $\tilde{U}$ we have that

$$\big(\lambda_r c(\lambda_r^{-1} w \lambda_{n_r})\big)(1) = c(\lambda_r^{-1} w \lambda_{n_r})(\lambda_r) = c(\lambda_r^{-1} w \lambda_{n_r})(1).$$

Therefore we deduce that

$$\sum_r^k \lambda_r c(\lambda_r^{-1} w \lambda_{n_r})(1) = \sum_r^k \delta_r c(\delta_r^{-1} w \delta_{n_r})(1).$$

□

One other important fact that tells us that we only have to look at Serre weights, that is to mean irreducible $\overline{\mathbb{F}}_p[\tilde{G}]$-modules for the analysis of Hecke eigenclasses is the following proposition.

Proposition 2.4.10. *Let $\mathfrak{n}$ be an integral ideal such that the positive generator of $\mathfrak{n} \cap \mathbb{Z}$ is greater than 3. Consider the open compact subgroup $K_1(\mathfrak{n})$ of level $\mathfrak{n}$. Let M be a finite dimensional $\overline{\mathbb{F}}_p[\tilde{G}]$-module. Let Ψ be a Hecke eigenvalue system with values in $\overline{\mathbb{F}}_p$ which occurs in $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), M)$. Then there exists an irreducible subquotient of M, say W, such that Ψ also occurs in $\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W)$.*

Proof. Let E be an irreducible submodule of M. Denote the quotient M/E as N. Set $K = K_1(\mathfrak{n})$. This is an open compact subgroup of $G(\hat{\mathcal{O}})$ which is neat and surjects onto $\hat{\mathcal{O}}^*$ via the determinant. Write the following exact sequence of locally constant sheaves on Y_K associated with E, M, N respectively:

$$0 \to \tilde{E} \to \tilde{M} \to \tilde{N} \to 0.$$

From this one obtains the exact sequence in cohomology:

$$\cdots \to \mathrm{H}^1(Y_K, \tilde{E}) \to \mathrm{H}^1(Y_K, \tilde{M}) \to \mathrm{H}^1(Y_K, \tilde{N}) \to \cdots .$$

Let s be a system of Hecke eigenvalues from $\mathrm{H}^1(Y_K, \tilde{M})$. If the image of s is zero, then s occurs in $\mathrm{H}^1(Y_K, \tilde{E})$, and we are done. Otherwise it is arisen from $\mathrm{H}^1(Y_K, \tilde{N})$. We then replace $\tilde{M}$ by $\tilde{N}$ and repeat the argument. $\square$

Statements and proofs of the main results

The statement about the reduction to weight two is as follows.

Theorem 2.4.11. *Let F be an imaginary quadratic field of class number h. Let $\mathfrak{n}$ be an integral ideal in F and let $p > 5$ be a rational prime which is inert in F and coprime with $\mathfrak{n}$. Suppose that the positive generator of $\mathfrak{n} \cap \mathbb{Z}$ is greater than 3. Let $0 \leq r, s \leq p-1$ and $0 \leq l, t \leq p-1$, with l, t not both equal to $p-1$. Let ψ be a system of Hecke eigenvalues in $\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p))$. Then ψ occurs in $\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p \otimes det^{l+pt})$ except possibly when $(r = 1, s = p-2)$ or $(r = p-2, s = 1)$. In these potential exceptions, the system of eigenvalues is Eisenstein.*

Proof. The proof is divided in two parts. Firstly, we show that

$$\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow \oplus_{i=1}^{h} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p)$$

as $\overline{\mathbb{F}}_p$-vector spaces except in the exceptional cases named in the statement. Secondly from this, we use an inflation restriction exact sequence and obtain an embedding of Hecke modules $\oplus_{i=1}^{h} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p)) \hookrightarrow \oplus_{i=1}^{h} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p \otimes det^{l+pt})$.

First part:

The exact sequence

$$0 \to V_{r,s}(\overline{\mathbb{F}}_p) \to \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p) \to W_{r,s} \to 0$$

gives rise to the long exact sequence in cohomology

$$\begin{aligned} 0 &\to \oplus_{i=1}^{h}\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \to \oplus_{i=1}^{h}\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)) \to \\ &\to \oplus_{i=1}^{h}\mathrm{H}^0(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W_{r,s}) \to \oplus_{i=1}^{h}\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \to \\ &\to \oplus_{i=1}^{h}\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)) \to \oplus_{i=1}^{h}\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), W_{r,s}) \to \cdots . \end{aligned}$$

This is an exact sequence of $\overline{\mathbb{F}}_p$-vector spaces. If $r = s = p-1$, from Lemmas 2.4.4, 2.4.5 and 2.4.6, we get the exact sequence of $\overline{\mathbb{F}}_p$-vector spaces for each $i = 1, \cdots, h$:

$$0 \to \overline{\mathbb{F}}_p \to \overline{\mathbb{F}}_p \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)) \to \cdots .$$

This means that the third arrow is the null map and hence we have an injection

$$\oplus_{i=1}^{h}\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow \oplus_{i=1}^{h}\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)).$$

From Lemmas 2.4.4, 2.4.5 and 2.4.6, we see that when ($r \neq 1$ or $s \neq p-2$) and ($r \neq p-2$ or $s \neq 1$), we have an exact sequence of $\overline{\mathbb{F}}_p$-vector spaces

$$0 \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)) \to \cdots .$$

Therefore in all cases this is an exact sequence of $\overline{\mathbb{F}}_p$-vector spaces. From Proposition 2.3.4, we know that the representation $\mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)$ is a direct summand of $\mathrm{Ind}_{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$. So, one has an embedding of $\overline{\mathbb{F}}_p$-vector spaces

$$\oplus_{i=1}^{h}\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{V}_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow \oplus_{i=1}^{h}\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{Ind}_{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)).$$

By Shapiro's lemma, one concludes that we have an injection of $\overline{\mathbb{F}}_p$-vector spaces

$$\alpha : \oplus_{i=1}^{h}\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow \oplus_{i=1}^{h}\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn})), \overline{\mathbb{F}}_p).$$

Lastly when ($r = 1, s = p-2$) or ($r = p-2, s = 1$), then from Lemmas 2.4.4, 2.4.5, 2.4.6 and 2.4.7, we have the exact sequence of $\overline{\mathbb{F}}_p$-vector spaces

$$0 \to \overline{\mathbb{F}}_p \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{U}_{r+ps}(\overline{\mathbb{F}}_p)) \to \cdots .$$

Second part:

Consider the inflation-restriction exact sequence

$$0 \to \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), (V^{l,t}_{r,s}(\overline{\mathbb{F}}_p))^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}) \xrightarrow{infl} \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) \xrightarrow{res}$$
$$\xrightarrow{res} \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p))^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})} \to \mathrm{H}^2(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}).$$

Because of the assumption concerning p we have that

$$\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), (V^{l,t}_{r,s}(\overline{\mathbb{F}}_p))^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}) = \mathrm{H}^2(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)^{\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}) = 0.$$

Then we get the isomorphism of $\overline{\mathbb{F}}_p$-vector spaces induced by the restriction map:

$$\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) \xrightarrow{\sim} (\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{0,0}_{r,s}(\overline{\mathbb{F}}_p)) \otimes_{\overline{\mathbb{F}}_p} det^{l+pt})^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n})}$$

where we have used the isomorphism

$$\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) \backsimeq \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{0,0}_{r,s}(\overline{\mathbb{F}}_p)) \otimes_{\overline{\mathbb{F}}_p} det^{l+pt}.$$

Next notice that for all $1 \leq i \leq h$, we have isomorphisms of abelian groups

$$\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}) \cong \mathcal{O}^* \cong \Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})/\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn}).$$

From the first part, when we are in the situation ($r \neq 1$ or $s \neq p-2$) and ($r \neq p-2$ or $s \neq 1$), then there is an embedding of $\overline{\mathbb{F}}_p$-vector spaces:

$$\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p).$$

When tensoring with det^{l+pt}, we obtain the embedding

$$\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p \otimes det^{l+pt}).$$

We next take $\mathcal{O}^*$-invariants and we get

$$(\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)))^{\mathcal{O}^*} \hookrightarrow (\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p \otimes det^{l+pt}))^{\mathcal{O}^*}.$$

This and the isomorphism induced by the inflation restriction exact sequence implies that

$$\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V^{l,t}_{r,s}(\overline{\mathbb{F}}_p)) \hookrightarrow (\mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{pn}), \overline{\mathbb{F}}_p \otimes det^{l+pt}))^{\mathcal{O}^*}.$$

Using once more the inflation restriction exact sequence for the right hand of this embedding, we derive that

$$\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p)) \hookrightarrow \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p \otimes det^{l+pt}).$$

This natural map is compatible with the Hecke action, and so this is an injection of Hecke modules.

Now when the cases $(r = 1, s = p-2)$ or $(r = p-2, s = 1)$ hold, then the first part provides us with an exact sequence

$$0 \to \overline{\mathbb{F}}_p \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}(\overline{\mathbb{F}}_p)) \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p).$$

This implies that the following sequences are exact:

$$0 \to \overline{\mathbb{F}}_p \otimes det^{l+pt} \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p)) \to \mathrm{H}^1(\Gamma^1_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p \otimes det^{l+pt})$$

$$\Rightarrow 0 \to (\overline{\mathbb{F}}_p \otimes det^{l+pt})^{\mathcal{O}^*} \to \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p)) \to \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p \otimes det^{l+pt}).$$

Then, when $(\overline{\mathbb{F}}_p \otimes det^{l+pt})^{\mathcal{O}^*} = 0$, the embedding

$$\mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), V_{r,s}^{l,t}(\overline{\mathbb{F}}_p)) \hookrightarrow \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n}), \overline{\mathbb{F}}_p \otimes det^{l+pt})$$

holds. Otherwise, we know that the obstruction is coming from $(\overline{\mathbb{F}}_p \otimes det^{l+pt})^{\mathcal{O}^*}$ and is hence Eisenstein as shown by Lemma 2.4.8. □

Systems of Hecke eigenvalues arising from $\overline{\mathbb{F}}_p \otimes det^e$ are Eisenstein and hence correspond to reducible Galois representations. Because of this, the statement about Serre's conjecture is not affected since it only concerns irreducible mod p Galois representations. Now the statement related to Serre type questions is as follows.

Proposition 2.4.12. *We keep the same conditions as in Theorem 2.4.11. A positive answer to question* (a) *on page 14 answers positively the question* (b) *and the reciprocal also holds.*

Proof. The part $(b) \Rightarrow (a)$ is obtained as follows. By Shapiro's lemma the system is realized in

$$\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), \mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{p}\mathfrak{n})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)).$$

By Proposition 2.4.10, this system of Hecke eigenvalues already appears in

$$\oplus_{i=1}^h \mathrm{H}^1(\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n}), M)$$

where M is a simple module from the Jordan-Hölder series of $\mathrm{Ind}_{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{pn})}^{\Gamma_{1,[\mathfrak{b}_i]}(\mathfrak{n})}(\overline{\mathbb{F}}_p)$. This module M is a Serre weight.

The part $(a) \Rightarrow (b)$ follows from Theorem 2.4.11. $\square$

Chapter 3

Hecke operators on Manin symbols

We describe Hecke operators on Manin symbols over imaginary quadratic fields of class number one.

3.1 Introduction

Let F be an imaginary quadratic field of class number one and with $\mathcal{O}$ as integer ring. For various reasons, it has become important to have a suitable computational approach for modular forms over F. The (co)homological point of view achieves this in a satisfactory way as it allows both theoretical and computational studies of modular forms. To the author's knowledge this approach was first taken in [19] where cusp forms over F were studied. Then Cremona [13] has carried out a computational study of weight two modular forms over the five euclidean imaginary quadratic fields of class number one. In chronological order, Whitley [41], Bygott [10], and Lingham [27] have extended Cremona's work to the four-non euclidean fields of class number one, fields of even class number, and fields of odd class number respectively. Let $\mathfrak{n}$ be an integral ideal of F and $\Gamma_0(\mathfrak{n})$ be the congruence subgroup defined as in the rational setting. In [10], [27] and [41], by a theorem from [26], weight two cusps forms are identified with the first homology $\mathrm{H}_1(\Gamma_0(\mathfrak{n})\backslash\mathbb{H}_3^*, \mathbb{C})$ as a Hecke module. The latter is computed via the modular symbols formalism.

The modular symbols formalism is a powerful tool, which enables one to gain theoretical and computational insights on modular forms and other related areas. As such, modular symbols for a congruence subgroup Γ, see Definition 3.2.6, are a quotient of two abelian groups of infinite rank, so apriori, they are not so suitable for concrete computation with computers. But, in fact, modular symbols admit a finite presentation. This is better seen via the theory of Manin symbols which provides an explicit basis for the space of modular symbols for Γ. Because of the computability of Manin symbols, modular symbols are computable and thus modular forms are so.

The importance of modular forms for arithmetical questions is mostly due to the fact that the information carried by modular forms can be read off via the action of Hecke operators. These Hecke operators act on the (co)homology of Γ with coefficients in a finite dimensional representation of Γ and on the space of modular symbols in a compatible fashion, that is the identification of the (co)homology with modular symbols respects the Hecke action. Unfortunately, a definition of Hecke operators on Manin symbols compatible with the one on modular symbols is not as straightforward as one could hope. In the classical setting the first description of Hecke operators on Manin symbols was done by Merel [30]. These Hecke operators are defined via the so-called Heilbronn matrices. Previously, authors had to convert Manin symbols to modular symbols in order to compute the Hecke action and then convert back to Manin symbols. So knowing how to define Hecke operators on Manin symbols intrinsically should ease computation with modular forms since there is no need to perform the time consuming ping-pong between Manin symbols and modular symbols.

Let R be an $\mathcal{O}$-module and consider V to be an $R[G]$-left module. Let $\mathcal{M}_R(\Gamma, V)$ be the space of modular symbols of weight V and level Γ, see below for the precise definition. Denote by $M_R(\Gamma, V)$ the space of Manin symbols for Γ and of weight V, see Definition 3.2.6. We will prove that we have the following commutative diagram of Hecke modules:

$$\begin{tikzcd} M_R(\Gamma, V) \arrow[r] \arrow[d, "\tilde{T}"] & \mathcal{M}_R(\Gamma, V) \arrow[d, "T"] \\ M_R(\Gamma, V) \arrow[r] & \mathcal{M}_R(\Gamma, V) \end{tikzcd}$$

where T, $\tilde{T}$ are Hecke operators and the horizontal arrows are surjective homomorphisms. The novelty being the definition of $\tilde{T}$ directly on Manin symbols such that the diagram commutes. Akin to the classical setting, the description of $\tilde{T}$ is explicit and does not depend on Γ.

To this end, in Section 3.2, we introduce the modular and Manin symbols formalism. In Section 3.3, we describe Hecke operators on Manin symbols.

3.2 Modular and Manin symbols

In this section we introduce the combinatorial definition of modular symbols and Manin symbols. The starting point is an exact sequence.

3.2.1 An exact sequence

Let F be an imaginary quadratic field of class number one and $\mathcal{O}$ its ring of integers. The following presentation is inspired from the one given in [39] where the classical case is treated. We set $G := \mathrm{SL}_2$ to be the special linear algebraic group over $\mathcal{O}$. We know that G is finitely generated, see [21]. There is a left action of G on the projective line over F, $\mathbb{P}^1(F)$, defined as: for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $z \in \mathbb{P}^1(F)$, $g.z := \frac{az+b}{cz+d}$. This action is transitive because F has class number one. We fix a set of generators of G once and for all as follows. We define $T_1, \cdots, T_l$ as generators of G_∞. The latter is the the stabilizer of ∞ in G for the action just described. Next we complete the set $\{T_1, \cdots, T_l\}$ with matrices $\sigma_1, \cdots, \sigma_r \in G$ such that $G =< \sigma_1, \cdots, \sigma_r, T_1, \cdots, T_l : Relations >$ where "Relations" stands for the relations among the σ_i and T_j.

Let R be an $\mathcal{O}$-module. In the proposition below $R[G]$ is viewed as a right $R[G]$-module.

Proposition 3.2.1. *The following sequence of R-modules*

$$\oplus_{i=1}^r R[G] \xrightarrow{(g_1,\cdots,g_r)\mapsto\sum_{i=1}^r g_i(1-\sigma_i)\infty} R[\mathbb{P}^1(F)] \xrightarrow{g.\infty\mapsto 1} R \to 0$$

is exact.

Proof. Since the action of G on $\mathbb{P}^1(F)$ is transitive we have a bijection

$$G/G_\infty \longleftrightarrow \mathbb{P}^1(F)$$

where g is mapped to $g.\infty$. This implies an isomorphism of R-modules

$$R[G]/I_\infty \to R[\mathbb{P}^1(F)]$$

with I_∞ the ideal of $R[G]$ generated by $1-h$ for $h \in G_\infty$. Now the augmentation homomorphism $R[G] \to R$ has kernel generated by $1-\sigma_1, \cdots, 1-\sigma_r, 1-T_1, \cdots, 1-T_l$. Let us denote the first arrow in the sequence by f, and by A the augmentation ideal. Next $Im(f) = \{\sum_{i=1}^r g_i(1-\sigma_i)\infty : (g_1 \cdots, g_r) \in \oplus_{i=1}^r R[G]\} = J.\infty$ where J is the $R[G]$-

module generated by $< 1-\sigma_1, \cdots, 1-\sigma_r >$. From the exact sequence

$$0 \to A \to R[G] \to R \to 0$$

we get the exactness of the sequence in the proposition at $R[\mathbb{P}^1(F)]$ by dividing out by I_∞. Indeed A/I_∞ is an $R[G]$-module which is generated by $1-\sigma_1, \cdots, 1-\sigma_r$, and we have an exact sequence of R-modules:

$$A/I_\infty \to (R[G]/I_\infty = R[\mathbb{P}^1(F)]) \to R \to 0.$$

This is what we needed.
The surjectivity of the second arrow is clear. $\square$

With this in hand, we next introduce the formalism of modular symbols and Manin symbols. The exact sequence in Proposition 3.2.1 will allow us to express the former in terms of the latter in a linear algebraic way as in the classical setting.

3.2.2 Modular and Manin symbols

Here F is still an imaginary quadratic field of class number one. The non-trivial automorphism of F is denoted by τ. For r, s integers we consider the representations of G :

$$V_{r,s}(\mathcal{O}) = \mathrm{Sym}^r(\mathcal{O}^2) \otimes_{\mathcal{O}} (\mathrm{Sym}^s(\mathcal{O}^2))^\tau = \mathcal{O}[X,Y]_r \otimes_{\mathcal{O}} \mathcal{O}[X,Y]_s^\tau,$$

where $\mathcal{O}[X,Y]_r$ is the space of homogeneous polynomials of degree r in the variables X, Y. Recall how this action is defined: for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and $P(X,Y) \otimes P'(X,Y) \in V_{r,s}(\mathcal{O})$, then, $g.(P(X,Y) \otimes P'(X,Y)) := P(\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)) \otimes P'(\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)^\tau \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)) = P(dX - bY, aY - cX) \otimes P'(d^\tau X + (-b)^\tau Y, a^\tau Y + (-c)^\tau X)$. For M an $\mathcal{O}$-module or an $\mathcal{O}$-algebra, we let $V_{r,s}(M) = V_{r,s}(\mathcal{O}) \otimes_{\mathcal{O}} M$. The modular symbols are defined as follows.

Definition 3.2.2. *Let R be an $\mathcal{O}$-algebra and V be a left $R[G]$-module.*

1. $\mathcal{M}_2 := R[\{\alpha,\beta\}]/ < \{\alpha,\alpha\}, \{\alpha,\beta\} + \{\beta,\gamma\} + \{\gamma,\alpha\}, \alpha,\beta \in \mathbb{P}^1(F) >$. *This is the R-module on the symbols $\{\alpha,\beta\}$, $\alpha,\beta \in \mathbb{P}^1(F)$ subject to the relations $\{\alpha,\alpha\} = \{\alpha,\beta\} + \{\beta,\gamma\} + \{\gamma,\alpha\} = 0$. This is called the space of weight 2 modular symbols over R.*

2. $\mathcal{B}_2 := R[\mathbb{P}^1(F)]$. *This is the space of weight 2 boundary symbols over R.*

3. $\mathcal{M}_R(V) := \mathcal{M}_2 \otimes_R V$. *This is called the space of modular symbols of weight V over R.*

4. $\mathcal{B}_R(V) := \mathcal{B}_2 \otimes_R V$. *This is the space of boundary symbols of weight* V *over* R.

5. *For* Γ *a congruence subgroup of* G, *the left coinvariants by the action of* Γ *on* $\mathcal{M}_R(V)$:

$$\mathcal{M}_R(\Gamma, V) := \mathcal{M}_R(V)/ < x - gx : g \in \Gamma, x \in \mathcal{M}_R(V) >$$

is called the space modular symbols of weight V *for* Γ, *over* R.

6. *Similarly the left coinvariants for the action of* Γ *on* $\mathcal{B}_R(V)$:

$$\mathcal{B}_R(\Gamma, V) := \mathcal{B}_R(V)/ < x - gx : g \in \Gamma, x \in \mathcal{B}_R(V) >$$

is called the space of boundary symbols of weight V *for* Γ, *over* R.

7. *The map* $\mathcal{M}_2 \to \mathcal{B}_2, \{\alpha, \beta\} \mapsto \beta - \alpha$ *called the boundary map, induces a map* $\mathcal{M}_R(\Gamma, V) \to \mathcal{B}_R(\Gamma, V)$ *also named the boundary map.*
   •*The kernel of the boundary map denoted by* $\mathcal{CM}_R(\Gamma, V)$ *is called the cuspidal modular symbols of level* Γ *and weight* V.
   •*The image of the boundary map denoted by* $\mathcal{E}_R(\Gamma, V)$ *is called the Eisenstein modular symbols of level* Γ *and weight* V.

In the above definitions, $g \in G$ acts on the symbol $\{y, z\}$ as: $g.\{y, z\} := \{g.y, g.z\}$. The action of g on $\mathcal{M}_R(V)$ and on $\mathcal{B}_R(V)$ is the diagonal action. If Γ has level $\mathfrak{n}$, then one can define modular symbols by twisting by a character $\chi : (\mathcal{O}/\mathfrak{n})^* \to R^*$, see [39] for details.

Lemma 3.2.3. *We have the following exact sequence of R-modules:*

$$0 \to \mathcal{M}_2 \xrightarrow{\{y,z\} \mapsto z-y} R[\mathbb{P}^1(F)] \xrightarrow{z \mapsto 1} R \to 0.$$

Proof. Because of the relation $\{y, z\} = \{\infty, z\} - \{\infty, y\}$, any element in $\mathcal{M}_2$ can be written as $\sum_{\omega \neq \infty} r_\omega \{\infty, \omega\}$. The latter under the first arrow has image $\sum_{\omega \neq \infty} r_\omega \omega - (\sum_{\omega \neq \infty} r_\omega)\infty$. And this is zero if and only if all the r_ω are zero.

For the exactness at the middle let $\sum_\omega r_\omega \omega \in R[\mathbb{P}^1(F)]$ be in the kernel of the augmentation map. Then we have $\sum_\omega r_\omega = 0$. Therefore we obtain

$$\begin{aligned}\sum_{\omega} r_\omega \omega &= \sum_{\omega} r_\omega \omega - (\sum_{\omega} r_\omega)\infty \\ &= \sum_{\omega\neq\infty} r_\omega \omega + r_\infty\infty - r_\infty\infty - (\sum_{\omega\neq\infty} r_\omega)\infty \\ &= \sum_{\omega\neq\infty} r_\omega \omega - (\sum_{\omega\neq\infty} r_\omega)\infty.\end{aligned}$$

The latter lies in the image of the first arrow. $\square$

From Proposition 3.2.1 and Lemma 3.2.3 we get the following statement.

Proposition 3.2.4. *The homomorphism of R-modules*

$$\begin{aligned}\phi : \oplus_{i=1}^{r} R[G] &\to \mathcal{M}_2 \\ (g_1, \cdots, g_r) &\mapsto \sum_{i=1}^{r} g_i\{\sigma_i.\infty, \infty\}\end{aligned}$$

is surjective.

Proof. We have the commutative diagram of R-modules

$$\begin{array}{ccc}\oplus_{i=1}^{r} R[G] & \xrightarrow{(g_1,\cdots,g_r)\mapsto\sum_{i=1}^{r} g_i(1-\sigma_i).\infty} & R[\mathbb{P}^1(F)] \\ \Big\downarrow \phi & & \Big\downarrow Id \\ \mathcal{M}_2 & \xrightarrow{\{y,z\}\mapsto z-y} & R[\mathbb{P}^1(F)].\end{array}$$

From Lemma 3.2.3 we have $\mathcal{M}_2 \cong ker(R[\mathbb{P}^1(F)] \to R)$, while, from Proposition 3.2.1 we have $\oplus_{i=1}^{r} R[G]/ker(\phi) \cong ker(R[\mathbb{P}^1(F)] \to R)$. $\square$

One of the consequences of Proposition 3.2.4 for modular symbols of weight V and level Γ over R is as follows. For left $R[G]$-modules V and M, we identify the $R[G]$-modules $(\oplus_{i=1}^{r} M) \otimes_R V$ with $\oplus_{i=1}^{r} M \otimes V$. In particular for a congruence subgroup Γ of G, G acts from the right on the quotient $\Gamma\backslash G$ and we get a right action of G on V as : $v.g := g^{-1}.v$. With this $\oplus_{i=1}^{r} R[\Gamma\backslash G] \otimes_R V$ is a right $R[G]$-module: $(g_1 \otimes v_1, \cdots, g_r \otimes v_r).g := (g_1 g \otimes g^{-1} v_1, \cdots, g_r g \otimes g^{-1} v_r)$. We define a right G-action on $\oplus_{i=1}^{r} R[G] \otimes_R V$ as : $(g_1 \otimes v_1, \cdots, g_r \otimes v_r).g := (g_1 g \otimes v_1, \cdots, g_r g \otimes v_r)$. There is also a natural left diagonal action of Γ on $\oplus_{i=1}^{r} R[G] \otimes_R V$. The homomorphism

$$\begin{aligned} \varphi : (\oplus_{i=1}^{r} R[G] \otimes_R V)_\Gamma &\to \oplus_{i=1}^{r} R[\Gamma\backslash G] \otimes_R V \\ (g_1 \otimes v_1, \cdots, g_r \otimes v_r) &\mapsto (g_1 \otimes g_1^{-1} v_1, \cdots, g_r \otimes g_r^{-1} v_r) \end{aligned}$$

defines an isomorphism of $R[G]$-modules

$$(\oplus_{i=1}^{r} R[G] \otimes_R V)_\Gamma \cong \oplus_{i=1}^{r} R[\Gamma\backslash G] \otimes_R V.$$

Indeed, it is well defined because for $\gamma \in \Gamma$, by denoting $\varphi((g_1 \otimes v_1 - \gamma(g_1 \otimes v_1), \cdots,$ $g_r \otimes v_r - \gamma(g_r \otimes v_r)))$ as $*$, then we have

$$\begin{aligned} * &= (g_1 \otimes g_1^{-1} v_1, \cdots, g_r \otimes g_r^{-1} v_r) - (\gamma g_1 \otimes g_1^{-1}\gamma^{-1}\gamma v_1, \cdots, \gamma g_r \otimes g_r^{-1}\gamma^{-1}\gamma v_r) \\ &= (g_1 \otimes g_1^{-1} v_1, \cdots, g_r \otimes g_r^{-1} v_r) - (\gamma g_1 \otimes g_1^{-1} v_1, \cdots, \gamma g_r \otimes g_r^{-1} v_r) = 0. \end{aligned}$$

We also have

$$\begin{aligned} \varphi((g_1 \otimes v_1, \cdots, g_r \otimes v_r).g) &= \varphi(g_1 g \otimes v_1, \cdots, g_r g \otimes v_r) \\ &= (g_1 g \otimes g^{-1} g_1^{-1}.v_1, \cdots, g_r g \otimes g^{-1} g_r^{-1} v_r) \\ &= (g_1 \otimes g_1^{-1}.v_1, \cdots, g_r \otimes g_r^{-1} v_r).g \\ &= \varphi(g_1 \otimes v_1, \cdots, g_r \otimes v_r).g. \end{aligned}$$

So φ is G-equivariant. Now one checks that the homomorphism

$$\begin{aligned} \varphi' : \oplus_{i=1}^{r} R[\Gamma\backslash G] \otimes_R V &\to (\oplus_{i=1}^{r} R[G] \otimes_R V)_\Gamma \\ (g_1 \otimes v_1, \cdots, g_r \otimes v_r) &\mapsto (g_1 \otimes g_1.v_1, \cdots, g_r \otimes g_r.v_r) \end{aligned}$$

defines the G-equivariant inverse of φ.

With regard to the next theorem, we view $\mathcal{M}_R(\Gamma, V)$ as a right $R[G]$-module: for $\gamma, g \in G$ and a modular symbol $g.\{\sigma_i.\infty, \infty\} \otimes P$; $(g.\{\sigma_i.\infty, \infty\} \otimes P).\gamma := g\gamma\{\sigma_i.\infty, \infty\} \otimes P$.

Theorem 3.2.5. *The homomorphism ϕ from Proposition 3.2.4 induces the exact sequence of right $R[G]$-modules*

$$\oplus_{i=1}^{r} R[\Gamma\backslash G] \otimes_R V \to \mathcal{M}_R(\Gamma, V) \to 0$$

where the homomorphism $\oplus_{i=1}^{r} R[\Gamma\backslash G] \otimes_R V \to \mathcal{M}_R(\Gamma, V)$ is given by $(g_1 \otimes v_1, \cdots, g_r \otimes v_r) \mapsto \sum_{i=1}^{r} g_i\{\sigma_i.\infty, \infty\} \otimes g_i v_i$.

Proof. Start with the right exact sequence of $R[G]$-modules coming from Proposition 3.2.4

$$\oplus_{i=1}^{r} R[G] \xrightarrow{\phi} \mathcal{M}_2 \to 0.$$

Apply the functor $. \otimes_R V$, which is right exact, to obtain the exact sequence of $R[G]$-modules

$$\oplus_{i=1}^{r} R[G] \otimes_R V \xrightarrow{\phi \otimes Id} \mathcal{M}_R(V) \to 0.$$

Now taking Γ-coinvariants is right exact and thus one has the exact sequence of $R[G]$-modules

$$(\oplus_{i=1}^{r} R[G] \otimes_R V)_\Gamma \to \mathcal{M}_R(\Gamma, V) \to 0.$$

Lastly we use the isomorphism of right $R[G]$-modules coming from page 57:

$$(\oplus_{i=1}^{r} R[G] \otimes_R V)_\Gamma \cong \oplus_{i=1}^{r} R[\Gamma \backslash G] \otimes_R V.$$

The resulting homomorphism is the composite of the following homomorphisms:

$$\oplus_{i=1}^{r} R[\Gamma \backslash G] \otimes_R V \xrightarrow{\varphi'} (\oplus_{i=1}^{r} R[G] \otimes_R V)_\Gamma \xrightarrow{\phi \otimes Id} \mathcal{M}_R(\Gamma, V)$$

with φ' the homomorphism defined in page 57. $\square$

We can now define what are called *Manin symbols.*

Definition 3.2.6. *1. We call an element in $\oplus_{i=1}^{r} R[G]$ a Manin symbol of weight 2 over R.*

2. An element in $\oplus_{i=1}^{r} R[G] \otimes_R V$ is called a Manin symbol of weight V over R.

3. An element in $\oplus_{i=1}^{r} R[\Gamma \backslash G] \otimes_R V$ is called a Manin symbol of weight V and level Γ over R. We denote this space as $M_R(\Gamma, V)$.

Remark 3.2.7. *When $r = 1$, the space of Manin symbols can be interpreted as an induced module. Recall that by definition the induced module $\mathrm{Ind}_\Gamma^G(V)$ is $R[G] \otimes_{R[\Gamma]} V$. This is a left $R[G]$-module with the natural action of G: $g \in G$ acts on the first factor by left multiplication. Now there is a natural diagonal action of Γ on $R[G] \otimes_R V$ and a right G-action given as : $(g \otimes v).g' = gg' \otimes v$. If we view $\mathrm{Ind}_\Gamma^G(V)$ as a right $R[G]$-module by inversion of the left action, then one has an isomorphism of right $R[G]$-modules:*

$$\mathrm{Ind}_\Gamma^G(V) \to (R[G] \otimes_R V)_\Gamma$$
$$g \otimes v \mapsto g^{-1} \otimes v.$$

This homomorphism is well defined because for $\gamma \in \Gamma, g\gamma \otimes v$ *is sent to* $\gamma^{-1}g^{-1} \otimes v = \gamma^{-1}(g^{-1} \otimes \gamma v)$ *but also* $g \otimes \gamma v$ *is sent to* $g^{-1} \otimes \gamma v$. *By definition* $g^{-1} \otimes \gamma v$ *and* $\gamma^{-1}(g^{-1} \otimes \gamma v)$ *are the same in* $(R[G] \otimes_R V)_\Gamma$. *Therefore from the isomorphism of* $R[G]$*-modules,* $(R[G] \otimes_R V)_\Gamma \cong R[\Gamma\backslash G] \otimes_R V$, *we deduce that* $\mathrm{Ind}_\Gamma^G(V)$ *is the space of Manin symbols of weight* V *and level* Γ *over* R.

Remark 3.2.8. *When describing Hecke operators on Manin symbols, the homomorphism provided by Theorem 3.2.5 will play a primordial role and by abuse of notation we will identify a Manin symbol with its image under that homomorphism in various places. So by abuse of notation, we verify the commutativity of the diagram in the introduction.*

By specializing the congruence subgroup, we can get a more explicit version of Manin symbols in the following way.

Manin symbols for $\Gamma_1(\mathfrak{n})$

Let $\mathfrak{n}$ be an ideal of $\mathcal{O}$. Consider the congruence subgroup $\Gamma_1(\mathfrak{n})$ of level $\mathfrak{n}$:

$$\Gamma_1(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G : c \equiv d - 1 \equiv 0 \pmod{\mathfrak{n}} \right\}.$$

Define $E_\mathfrak{n} = \left\{ (u,v) \in (\mathcal{O}/\mathfrak{n})^2 :< u, v >= \mathcal{O}/\mathfrak{n} \right\}$. We have a surjective map from G to $E_\mathfrak{n}$ which sends M to $(0,1)M$. As $\Gamma_1(\mathfrak{n})$ maps to $(0,1)$, then M, M' have the same image under this map if and only if $M \in \Gamma_1(\mathfrak{n})M'$. Hence we have a bijection:

$$\Psi : \Gamma_1(\mathfrak{n})\backslash G \longleftrightarrow E_\mathfrak{n}.$$

A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Mat_2(\mathcal{O})_{\neq 0}$ which has determinant coprime with $\mathfrak{n}$ acts on $E_\mathfrak{n}$ as:

$$(u,v).\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) := (u,v)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = (au + cv, bu + dv).$$

The space of Manin symbols of weight V and level $\Gamma_1(\mathfrak{n})$ over a ring R is then identified with $\oplus_{i=1}^r R[E_\mathfrak{n}] \otimes_R V$. In general the i-th entry of a Manin symbol in $M_R(\Gamma_1(\mathfrak{n}), V)$ is of the form $\sum_s (c_s, d_s) \otimes v_s$, and since the elements $(c,d) \otimes v$ generate this i-th entry, for our purpose it is enough to work with Manin symbols of the form

$$(0, \cdots, 0, \underbrace{(c,d) \otimes v}_{i\text{-}th\ component}, 0, \cdots, 0).$$

Then via the homomorphism given in Theorem 3.2.5, it corresponds to the modular symbol $g.\{\sigma_i.\infty, \infty\} \otimes g.v$ where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$.

Manin symbols for $\Gamma_0(\mathfrak{n})$

Let the ideal $\mathfrak{n}$ be as above. Let $\Gamma_0(\mathfrak{n})$ be the congruence subgroup of G defined as:

$$\Gamma_0(\mathfrak{n}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G : c \equiv 0 \pmod{\mathfrak{n}} \right\}.$$

We consider the projective line $\mathbb{P}^1(\mathcal{O}/\mathfrak{n})$ over the ring $\mathcal{O}/\mathfrak{n}$. Then the map from G to $\mathbb{P}^1(\mathcal{O}/\mathfrak{n})$ mapping $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ to $(c : d)$ induces a bijection

$$\Gamma_0(\mathfrak{n})\backslash G \longleftrightarrow \mathbb{P}^1(\mathcal{O}/\mathfrak{n}).$$

A matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ whose determinant is coprime with $\mathfrak{n}$ acts on $\mathbb{P}^1(\mathcal{O}/\mathfrak{n})$ as :

$$(e : f).M := (e : f)M = (ae + fc : eb + fd).$$

Hence the space of Manin symbols of weight V and level $\Gamma_0(\mathfrak{n})$ over R can be identified with $\oplus_{i=1}^{r} R[\mathbb{P}^1(\mathcal{O}/\mathfrak{n})] \otimes_R V$. Now the i-th entry $(c : d) \otimes v$ of the Manin symbol similar to the one defined in the end of Subsection 3.2.2 corresponds to the modular symbol $g.\{\sigma_i.\infty, \infty\} \otimes g.v$ with $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$.

We shall next turn to the description of Hecke operators on Manin symbols.

3.3 Hecke operators on Manin symbols

As the theory we have developed in Section 3.2 is valid when the quadratic field has class number one, in what follows the field F is of class number one. Here, we will start by describing Hecke operators on Manin symbols by generalizing Merel's approach to our setting and then give another description which generalizes Cremona's approach. In the classical setting Cremona's description of Hecke operators on Manin symbols is a simplification of Merel's original description, but, it does not unravel the connection between them. We shall see here that these two approaches are essentially the same.

3.3.1 Merel's approach

Let Γ be a congruence subgroup of G. Let Δ be a semi-subgroup of $Mat_2(\mathcal{O})$ such $\Delta\Gamma = \Gamma\Delta$ and the quotient $\Gamma\backslash\Delta$ is finite. Fix a set $\mathcal{R}$ of representatives of $\Gamma\backslash\Delta$. Then the Hecke operator T_Δ acting on $\mathcal{M}_R(\Gamma, V)$ is the linear map defined by

$$\begin{aligned} T_\Delta : \mathcal{M}_R(\Gamma, V) &\to \mathcal{M}_R(\Gamma, V) \\ \{\alpha, \beta\} \otimes P &\mapsto \sum_{\lambda \in \mathcal{R}} \{\lambda.\alpha, \lambda.\beta\} \otimes \lambda.P. \end{aligned}$$

Because of the definition of modular symbols, we see that the Hecke operator T_Δ does not depend on the set $\mathcal{R}$.

Let ι be the Shimura involution on the semi-group $Mat_2(\mathcal{O})_{\neq 0}$ of matrices with non-zero determinant and entries in $\mathcal{O}$ defined by:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^\iota := \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right).$$

For $g \in Mat_2(\mathcal{O})_{\neq 0}$, then $g^\iota = g^{-1} det(g)$. Via ι we get an action of the semi-group $Mat_2(\mathcal{O})_{\neq 0}$ on $V_{r,s}(\mathcal{O})$ extending the action of G defined in Subsection 3.2.1:

$$g.P(X, Y) := P(g^\iota \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)).$$

Next define $\Delta^\iota = \{g \in Mat_2(\mathcal{O})_{\neq 0} : g^\iota \in \Delta\}$. Furthermore we suppose that the following conditions hold.

1. There exists a map $\psi : \Delta^\iota G \to G$ such that
   (a) for all $\gamma \in \Delta^\iota G$ and $g \in G$ we have $\Gamma\psi(\gamma g) = \Gamma\psi(\gamma)g$ and
   (b) for all $\gamma \in \Delta^\iota G$ we have $\gamma\psi(\gamma)^{-1} \in \Delta^\iota$.
2. We require that the map $\Gamma\backslash\Delta \to \Delta^\iota G/G$ where $\Gamma\lambda$ is sent to $\lambda^\iota G$ is a bijection.

These conditions will ensure that summing over $g\Delta^\iota G$ is the same as summing over the quotient $\Gamma\backslash\Delta$.

Recall that we have fixed a set of generators of G in Subsection 3.2.1, and the matrices σ_i are the matrices in that set not stabilizing ∞. The number of these σ_i is denoted by r.

Definition 3.3.1. *For all $1 \leq i, j \leq r$ and all $\theta \in Mat_2(\mathcal{O})_{\neq 0}$, let $a_{i,j,\theta} \in R$. The collection $\{a_{i,j,\theta} : 1 \leq i, j \leq r, \theta \in Mat_2(\mathcal{O})_{\neq 0}\}$ satisfies Merel's condition C_Δ if and only if for all i and for all classes $K \in \Delta^\iota G/G$, the following equality in $\mathcal{M}_2$ holds*

$$\sum_{j=1}^{r} \sum_{\theta \in K} a_{i,j,\theta}\theta\{\sigma_j.\infty, \infty\} = \{\sigma_i.\infty, \infty\}.$$

Recall that a Manin symbol of weight V and level Γ is an element in $\oplus_{i=1}^{r} R[\Gamma\backslash G]\otimes V$. A Manin symbol $(0,\cdots,0,g\otimes P,0,\cdots,0)$ with $g\otimes P$ at the i-th entry corresponds to the modular symbol $g(\{\sigma_i.\infty,\infty\}\otimes P)$.

Theorem 3.3.2. *For $0\leq i,j\leq r$ and $\theta\in Mat_2(\mathcal{O})_{\neq 0}$, let $a_{i,j,\theta}\in R$ satisfy Merel's C_Δ condition. Then the Hecke operator T_Δ on the Manin symbol $(0,\cdots,0,g\otimes P,0,\cdots,0)\in\oplus_{i=1}^{r} R[\Gamma\backslash G]\otimes V$ with $g\otimes P$ in the i-th entry has j-th entry given as*

$$(T_\Delta.(0,\cdots,0,g\otimes P,0,\cdots,0))_j=\sum_{\{\theta:\theta,g\theta\in\Delta^\iota G\}} a_{i,j,\theta}\psi(g\theta)\otimes\theta^\iota.P.$$

Proof. As the Manin symbol $(0,\cdots,0,g\otimes P,0,\cdots,0)$ corresponds to the modular symbol $g(\{\sigma_i.\infty,\infty\}\otimes P)$, we have to show that

$$\begin{aligned}T_\Delta.(g\{\sigma_i.\infty,\infty\}\otimes g.P) &= \sum_{\lambda\in\Gamma\backslash\Delta}\lambda(g\{\sigma_i.\infty,\infty\}\otimes g.P)\\ &= \sum_{\theta\in g^{-1}\Delta^\iota G}\sum_{j=1}^{r} a_{i,j,\theta}\psi(g\theta)\{\sigma_j.\infty,\infty\}\otimes\psi(g\theta)\theta^\iota.P.\end{aligned}$$

Let $(*)$ denote the last sums in the equation we wish to establish. Let $\mathcal{R}$ be a set of representatives of $g^{-1}\Delta^\iota G/G$. So we write the latter as a disjoint union of left cosets $\alpha G: g^{-1}\Delta^\iota G=\amalg_{\alpha\in\mathcal{R}}\alpha G$. Then we write

$$\begin{aligned}(*) &= \sum_{\alpha\in\mathcal{R}}\sum_{\theta\in\alpha G}\sum_{j=1}^{r} a_{i,j,\theta}\psi(g\alpha\alpha^{-1}\theta)\{\sigma_j.\infty,\infty\}\otimes\psi(g\alpha\alpha^{-1}\theta)\theta^\iota.P\\ &= \sum_{\alpha\in\mathcal{R}}\sum_{\theta\in\alpha G}\sum_{j=1}^{r} a_{i,j,\theta}\psi(g\alpha)\alpha^{-1}\{\theta.\sigma_j\infty,\theta.\infty\}\otimes\psi(g\alpha)\alpha^{-1}\theta\theta^\iota.P.\end{aligned}$$

The last equality is because of property 1. (a) : $\Gamma\psi(\gamma g)=\Gamma\psi(\gamma)g\ \forall\ \gamma\in\Delta^\iota G, g\in G$. Because $\theta\in\alpha G$, we have that $det(\theta)=det(\alpha)$ and hence $\alpha^{-1}\theta\theta^\iota=\alpha^\iota$. Now denote $\sum_{j=1}^{r}\sum_{\theta\in\alpha G} a_{i,j,\theta}\psi(g\alpha)\alpha^{-1}\{\theta.\sigma_j.\infty,\theta.\infty\}\otimes\psi(g\alpha)\alpha^{-1}\theta\theta^\iota P$ as $*$, then by condition C_Δ we have

$$* = \psi(g\alpha)\alpha^{-1}\{\sigma_i.\infty,\infty\}\otimes\psi(g\alpha)\alpha^\iota.P.$$

Therefore by modular symbols properties, namely the fact that scalars act trivially on modular symbols, and by denoting $\sum_{j=1}^{r}\sum_{\theta\in g^{-1}\Delta^\iota G} a_{i,j,\theta}\psi(g\theta)(\{\sigma_j.\infty,\infty\}\otimes\theta^\iota.P)$ as $*_1$, one has

$$\begin{aligned} *_1 &= \sum_{\alpha\in\mathcal{R}} \psi(g\alpha)\alpha^{-1}\{\sigma_i.\infty,\infty\}\otimes\psi(g\alpha)\alpha^\iota.P \\ &= \sum_{\alpha\in\mathcal{R}} \psi(g\alpha)\alpha^\iota g^\iota g\{\sigma_i.\infty,\infty\}\otimes\psi(g\alpha)\alpha^\iota g^\iota g.P. \end{aligned}$$

From property 1. (b), we know that $\psi(g\alpha)\alpha^\iota g^\iota$ lies in Δ. From the bijection $\Gamma\backslash\Delta\to\Delta^\iota G/G$, it follows that the right cosets $\Gamma\psi(g\alpha)\alpha^\iota g^\iota$ and $\Gamma\psi(g\alpha_1)\alpha_1^\iota g^\iota$ are disjoint when $\alpha\neq\alpha_1$. This means that $\psi(g\alpha)\alpha^\iota g^\iota$ for $\alpha\in\mathcal{R}$ is a set of representatives of $\Gamma\backslash\Delta$. Therefore

$$\begin{aligned} \sum_{j=1}^{r}\sum_{\theta\in g^{-1}\Delta^\iota G} a_{i,j,\theta}\psi(g\theta)(\{\sigma_j.\infty,\infty\}\otimes\theta^\iota.P) &= \sum_{\lambda\in\Gamma\backslash G}\lambda(g\{\sigma_i\infty,\infty\}\otimes g.P) \\ &= T_\Delta.(g\{\sigma_i\infty,\infty\}\otimes g.P). \end{aligned}$$

Now translating the left hand side of the above equalities into Manin symbols by using the correspondence between Manin symbols and modular symbols, we obtain that the j-th component of $T_\Delta(0,\cdots,0,g\otimes P,0,\cdots,0)$ is as claimed. This ends the proof of the theorem. □

We shall specialize to $\Gamma_1(\mathfrak{n})$.

Hecke action on Manin symbols of level $\Gamma_1(\mathfrak{n})$

As one application of Theorem 3.3.2, we shall describe the Hecke action on Manin symbols of weight V and level $\Gamma_1(\mathfrak{n})$ where $\mathfrak{n}$ is an ideal of $\mathcal{O}$. Let $\eta\in\mathcal{O}$ be non-zero. We define Δ_η by

$$\Delta_\eta = \left\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in Mat_2(\mathcal{O}) : ad-bc=\eta, c\equiv a-1\equiv 0 \pmod{\mathfrak{n}}\right\}.$$

Then $\Gamma_1(\mathfrak{n})\Delta_\eta=\Delta_\eta\Gamma_1(\mathfrak{n})$ and $\Gamma_1(\mathfrak{n})\backslash\Delta_\eta$ is finite. Next we write T_η as the linear operator T_{Δ_η} on $\mathcal{M}_R(\Gamma_1(\mathfrak{n}),V)$. Let a map

$$\phi_\eta : \Delta_\eta^\iota G\to G$$

be such that $\Psi(\phi_\eta(M))=(0,1)M\in E_\mathfrak{n}$ with Ψ the bijection defined in Subsection 3.2.2. Explicitly an element $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Delta_\eta^\iota G$ is sent to $\left(\begin{smallmatrix} a' & b'\\ c & d\end{smallmatrix}\right)\in G$ under ϕ_η.

Lemma 3.3.3. *The map ϕ_η and Δ_η satisfy conditions* 1 *and* 2 *from page 61.*

Proof. First we need to verify for all $\gamma\in\Delta_\eta^\iota G$ and $g\in G$ we have $\Gamma_1(\mathfrak{n})\phi_\eta(\gamma g)=$

$\Gamma_1(\mathfrak{n})\phi_\eta(\gamma)g$. So take $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Delta_\eta^\iota$ and $g \in G$. Then one has

$$\Psi(\phi_\eta(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) g)) = (c,d)g = \Psi(\phi_\eta(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right))g).$$

Hence $\Gamma_1(\mathfrak{n})\phi_\eta(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) g) = \Gamma_1(\mathfrak{n})\phi_\eta(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)).g$.

Next we verify that for all $\gamma \in \Delta_\eta^\iota G$ we have $\gamma\phi_\eta(\gamma)^{-1} \in \Delta_\eta^\iota$. First observe that an element $g \in \Delta_\eta^\iota G$ is also in Δ_η^ι if and only if $(0,1)g = (0,1)$ modulo $\mathfrak{n}$. Then for $g \in \Delta_\eta^\iota G$ we have $(0,1)g = (0,1)\phi_\eta(g)$ and hence $g\phi_\eta(g)^{-1} \in \Delta_\eta^\iota$ as wanted.

For the second condition we need to check that there is a bijection $\Gamma_1(\mathfrak{n})\backslash\Delta_\eta \to \Delta_\eta^\iota G/G$ where $\Gamma_1(\mathfrak{n})\lambda$ is mapped to $\lambda^\iota G$. This map is surjective so we have to verify injectivity. Take $\delta = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \delta' = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in \Delta_\eta$ such that $\delta^\iota G = \delta'^\iota G$. The latter equality implies that

$$\delta'\delta^{-1} = \frac{1}{\eta}\left(\begin{smallmatrix} da'-b'c & -ba'+ab' \\ dc'-d'c & -bc'+ad' \end{smallmatrix}\right) \in G.$$

From the definition of Δ_η one deduces that $\delta'\delta^{-1} \in \Gamma_1(\mathfrak{n})$ and thus we have injectivity. $\square$

Next we introduce the condition C_η which is a variant of the condition C_{Δ_η} as follows.

Definition 3.3.4. *Let $Mat_2(\mathcal{O})_\eta$ be the set of matrices with determinant η. For all $1 \leq i,j \leq r$ and all $\theta \in Mat_2(\mathcal{O})_\eta$, let $a_{i,j,\theta} \in R$. The collection $\{a_{i,j,\theta} : 1 \leq i,j \leq r, \theta \in Mat_2(\mathcal{O})_\eta\}$ satisfies condition C_η if and only if for all i and for all classes K in $Mat_2(\mathcal{O})_\eta/G$ the following holds*

$$\sum_{j=1}^{r}\sum_{\theta\in K} a_{i,j,\theta}\theta\{\sigma_j.\infty,\infty\} = \{\sigma_i.\infty,\infty\}.$$

From Subsection 3.2.2, a Manin symbol over R of weight V and level $\Gamma_1(\mathfrak{n})$ is an element in $\oplus_{i=1}^r R[E_\mathfrak{n}]\otimes_R V$. A matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Mat_2(\mathcal{O})$ with determinant κ coprime with $\mathfrak{n}$ acts on $E_\mathfrak{n}$ as:

$$(u,v).M := (u,v)M = (au+vc, ub+vd).$$

We consider the Manin symbol $(0,\cdots,0,(u,v)\otimes P,0,\cdots,0)$ with $(u,v)\otimes P$ at the i-th entry and 0 otherwise. We know that it corresponds to the modular symbol $\left(\begin{smallmatrix} x & y \\ u & v \end{smallmatrix}\right)(\{\sigma_i.\infty,\infty\}\otimes P)$, where $x,y \in \mathcal{O}$ are such that $\left(\begin{smallmatrix} x & y \\ u & v \end{smallmatrix}\right) \in G$.

Proposition 3.3.5. *For all $1 \leq i,j \leq r$ and $\theta \in Mat_2(\mathcal{O})_\eta$, let $a_{i,j,\theta} \in R$ satisfy condition C_η. Then the Hecke operator T_η on $(0,\cdots,0,(u,v)\otimes P,0,\cdots,0)$ with $(u,v)\otimes P$*

in the i-entry has j-entry given by

$$(T_\eta.(0,\cdots,0,(u,v)\otimes P,0,\cdots,0))_j = \sum_{\theta\in g^{-1}\Delta_\eta^\iota G} a_{i,j,\theta}(u,v)\theta\otimes\theta^\iota.P.$$

Proof. Since the set of classes of $\Delta_\eta^\iota G/G$ is a subset of the set of classes of $Mat_2(\mathcal{O})_\eta/G$, we deduce that condition C_η implies condition C_{Δ_η}. Now we can apply Theorem 3.3.2 with ϕ_η and Δ_η^ι. Let $g\in G$ correspond to (u,v), i.e, $g=\left(\begin{smallmatrix} x & y\\ u & v\end{smallmatrix}\right)$. Then from Theorem 3.3.2 we have

$$\begin{aligned}(T_\eta(0,\cdots,0,(u,v)\otimes P,0,\cdots,0))_j &= (T_{\Delta_\eta}(0,\cdots,0,g\otimes P,0,\cdots,0))_j\\ &= \sum_{\theta\in g^{-1}\Delta_\eta^\iota G} a_{i,j,\theta}\phi_\eta(g\theta)\otimes\theta^\iota.P\\ &= \sum_{\theta\in g^{-1}\Delta_\eta^\iota G} a_{i,j,\theta}(u,v)\theta\otimes\theta^\iota.P\end{aligned}$$

where we used the fact that $\phi_\eta(g\theta)$ has bottom row $(u,v)\theta$ and the Manin symbol $\phi_\eta(g\theta)\otimes\theta^\iota.P$ corresponds to the Manin symbol $(u,v)\theta\otimes\theta^\iota.P$. □

We shall next make some observations concerning condition C_η. More precisely we will write down a set of representatives for $Mat_2(\mathcal{O})_\eta/G$, and derive what shape condition C_η takes. Let $\mathcal{D}$ be the set of divisors of η where we identify two divisors when they are associate. That is, if δ and δ' are distinct elements in $\mathcal{D}$ then $\delta/\delta'\notin\mathcal{O}^*$. The set $\mathcal{D}$ will called a set of "positive" divisors of η.

Lemma 3.3.6. *Let $\mathcal{D}$ be a set of positive divisors of η. The set R_η of matrices*

$$\left\{\begin{pmatrix}\delta & \beta\\ 0 & \eta/\delta\end{pmatrix} : \delta\in\mathcal{D},\beta \text{ running through a set of representatives for } \mathcal{O}/\delta\mathcal{O}\right\}$$

is a set of representatives of $Mat_2(\mathcal{O})_\eta/G$.

Proof. A matrix $g\in Mat_2(\mathcal{O})_\eta$ belongs to a class modulo G of an upper right triangular matrix. Indeed, since G acts transitively on $\mathbb{P}^1(F)$, there is a matrix $\omega\in G$ such that $g\omega.\infty=\infty$. Hence $g\omega$ is upper triangular and g belongs to its class. Now two matrices of the form $\begin{pmatrix}\delta & \beta\\ 0 & \eta/\delta\end{pmatrix}$ and $\begin{pmatrix}\delta' & \beta'\\ 0 & \eta/\delta'\end{pmatrix}$ belong to the same class modulo G if and only if $\begin{pmatrix}\delta' & \beta'\\ 0 & \eta/\delta'\end{pmatrix}^{-1}\begin{pmatrix}\delta & \beta\\ 0 & \eta/\delta\end{pmatrix} = \begin{pmatrix}\delta/\delta' & \beta/\delta'-\beta'/\delta\\ 0 & \delta'/\delta\end{pmatrix}\in G$. That is, if and only if $\delta=\delta'$ and $\beta\equiv\beta'$ (mod δ). □

Let $a_{i,j,\theta}\in R$ satisfy condition C_η. A class $K\in Mat_2(\mathcal{O})_\eta/G$ is given as $K=\gamma G$

where $\gamma \in R_\eta$. For each $\theta \in K$ there exists $g_\theta \in G$ such that $\theta = \gamma g_\theta$. Now condition C_η becomes

$$\sum_{j=1}^{r} \sum_{\theta \in K} a_{i,j,\theta} \gamma g_\theta \{\sigma_j.\infty, \infty\} = \{\sigma_i.\infty, \infty\}.$$

Applying to the above equation the matrix γ^{-1} and using the fact that scalars acts trivially on modular symbols, we obtain that

$$\sum_{j=1}^{r} \sum_{\theta \in K} a_{i,j,\theta} g_\theta \{\sigma_j.\infty, \infty\} = \{\gamma^{-1}\sigma_i.\infty, \infty\}.$$

Therefore condition C_η amounts to writing the modular symbols $\{\gamma^{-1}\sigma_i.\infty, \infty\}$ as linear combination of the modular symbols $g\{\sigma_j.\infty, \infty\}$ with $g \in G$. At this point a comment about the nomenclature of the elements $g\{\sigma_j.\infty, \infty\}$ for $g \in G$ is in order. In the classical setting and in ours for previous authors, these elements are called Manin symbols, so, there is a discrepancy as what we have named Manin symbols differ from the usage in the literature. However we saw that our Manin symbols surject on the space of modular symbols and the homomorphism is given in terms of classical Manin symbols. Thus from Theorem 3.2.5 , any modular symbol can be written as a linear combination of modular symbols of type $g(\{\sigma_j.\infty, \infty\} \otimes P)$. Classically this fact is established by using continued fractions, the so-called Manin's trick.

In the spirit of Manin's trick, we can see that the modular symbol $\{\alpha, \infty\} \otimes P$ can be written as a linear combination of modular symbols of type $g(\{\sigma_j.\infty, \infty\} \otimes g^{-1}.P)$ as follows.

Proposition 3.3.7. *Let G be given as $G =< \sigma_1, \cdots, \sigma_r; T_1, \cdots, T_l : Relations >$, with $G_\infty =< T_1, \cdots, T_l >$, the stabilizer of ∞ for the action of G on $\mathbb{P}^1(F)$. Let $\alpha \in \mathbb{P}^1(F)$ and take $g \in G$ such that $g.\infty = \alpha$ which is always possible since G acts transitively on $\mathbb{P}^1(F)$. Write g as $g = \sigma_1^{a_{1,1}} T_2^{b_{2,1}} \sigma_r^{a_{r,1}} T_l^{b_{l,1}} \cdots T_1^{b_{1,k}} \sigma_r^{a_{r,s}}$ for instance. One has*

1. $\{\alpha, \infty\} = \sigma_1^{a_{1,1}} T_2^{b_{2,1}} \sigma_r^{a_{r,1}} T_l^{b_{l,1}} \cdots T_1^{b_{1,k}} \sigma_r^{a_{r,s}-1}\{\sigma_r.\infty, \infty\} + \cdots + \sigma_1^{a_{1,1}-1}\{\sigma_1.\infty, \infty\} + \cdots + \{\sigma_1.\infty, \infty\}$.

2. *Let* $\{\alpha, \infty\} = \sum_{j=1}^{r} \sum_{k=1}^{s_j} L_{j,k}\{\sigma_j.\infty, \infty\}$ *with* $L_{j,k} \in G$. *Then*

$$\{\alpha, \infty\} \otimes P = \sum_{j=1}^{r} \sum_{k=1}^{s_j} L_{j,k}(\{\sigma_j.\infty, \infty\} \otimes L_{j,k}^{-1}.P).$$

Proof. From the definition of the modular symbols, and by denoting

$$\sigma_1^{a_{1,1}} T_2^{b_{2,1}} \sigma_r^{a_{r,1}} T_l^{b_{l,1}} \cdots T_1^{b_{1,k}} \sigma_r^{a_{r,s}-1}$$

as ζ, we can write

$$\{\alpha, \infty\} = \zeta\{\sigma_r.\infty, \infty\} + \{\zeta.\infty, \infty\}.$$

Do the same with $\{\sigma_1^{a_{1,1}} T_2^{b_{2,1}} \sigma_r^{a_{r,1}} T_l^{b_{l,1}} \cdots T_1^{b_{1,k}} \sigma_r^{a_{r,s}-1}.\infty, \infty\}$ and so on.

As for the second item we have

$$\{\alpha, \infty\} \otimes P = \sum_{j=1}^{r} \sum_{k=1}^{s_j} L_{j,k}\{\sigma_j.\infty, \infty\} \otimes P = \sum_{j=1}^{r} \sum_{k=1}^{s_j} L_{j,k}(\{\sigma_j.\infty, \infty\} \otimes L_{j,k}^{-1}.P).$$

□

For applying Proposition 3.3.7, two questions need to be dealt with, namely for $\alpha \in \mathbb{P}^1(F)$ how to find $g \in G$ such that $g.\infty = \alpha$ and how to obtain the word decomposition of g. The latter question is some how less easy as usually for non euclidean fields one makes use of a fundamental domain of the action of G on the 3-dimensional hyperbolic space $\mathcal{H}_3 = \mathbb{C} \times \mathbb{R}_{>0}$. In the case of fields with an euclidean algorithm, we will give an effective algorithm which solves the problem. The former question is some how easy to address. One way of finding such a g for a given α is as follows.

Algorithm 3.3.8. Input*:* $\alpha = \frac{a}{c}$ *in lowest terms.*
Output*:* $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$.

1. *Solve for* $\bar{d}$ *the equation* $\bar{a}\bar{d} \equiv 1 \pmod{c}$[1].

2. *Take some lift* d *of* $\bar{d}$ *to* $\mathcal{O}$ *and set* $b := \frac{ad-1}{c} \in \mathcal{O}$.

3. *Return* $g := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

Going back to condition C_η, we just saw that this condition amounts to write down a modular symbol as linear combination of certain special modular symbols. As we know this is always possible, and perhaps this is the observation that has led to a less

[1] In the euclidean case, the extended euclidean algorithm can be used to find such a d. In the non-euclidean class number one case, there is an extended euclidean algorithm for Dedekind domains by Cohen, see [11]. Alternatively, as we are dealing with finite rings, a search can be performed when the norm of c is reasonable.

complicated description of Hecke operator on Manin symbols as we shall next see. In the rational case, the description is due to Cremona [14].

3.3.2 Hecke operators à la Cremona

Here we will give a less involved description of Hecke operators on Manin symbols of weight V and level $\Gamma_0(\mathfrak{n})$ over R. In the end, we shall present an algorithm which computes these Hecke operators for both descriptions, i.e, the one above and the forthcoming. Unfortunately this algorithm is only easily described in the euclidean case. In the non-euclidean case or higher class number, more machinery is needed. As alluded to, the starting point is the expansion of a modular symbol $\{\alpha, \infty\}$ as a linear combination of modular symbols of type $g\{\sigma_j.\infty, \infty\}$.

Hecke operators

Let $\mathfrak{n}$ be a non-zero ideal in $\mathcal{O}$. We take a non-zero element η in $\mathcal{O}$ and coprime with $\mathfrak{n}$. Let R_η be the set of representatives of $Mat_2(\mathcal{O})_\eta/G$ given by Lemma 3.3.6. Let $\Gamma_0(\mathfrak{n}) = \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G : c \equiv 0 \pmod{\mathfrak{n}}\}$. Let $\mathbb{P}^1(\mathcal{O}/\mathfrak{n})$ be the projective line over $\mathcal{O}/\mathfrak{n}$. We know from Subsection 3.2.2 that a Manin symbol of weight V and level $\Gamma_0(\mathfrak{n})$ lives in $\oplus_{i=1}^r R[\mathbb{P}^1(\mathcal{O}/\mathfrak{n})] \otimes_R V$. We also know that the Manin symbol $(0, \cdots, 0, (c:d)\otimes P, 0, \cdots, 0)$ with $(c:d)\otimes P$ in the i-th entry corresponds to the modular symbol $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right).(\{\sigma_i.\infty, \infty\}\otimes P)$ for any $a, b \in \mathcal{O}$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. We shall need the following lemma.

Lemma 3.3.9. *Let $(c:d) \in \mathbb{P}^1(\mathcal{O}/\mathfrak{n})$ with c, d, coprime and let $\begin{pmatrix} \delta & \beta \\ 0 & \eta/\delta \end{pmatrix} \in R_\eta$. Then, there exist $a, b \in \mathcal{O}$ and a matrix $\begin{pmatrix} \delta' & \beta' \\ 0 & \eta/\delta' \end{pmatrix} \in R_\eta$ such that*

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \begin{pmatrix} \delta & \beta \\ 0 & \eta/\delta \end{pmatrix} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) (\begin{pmatrix} \delta' & \beta' \\ 0 & \eta/\delta' \end{pmatrix}^\iota)^{-1} \in G.$$

Proof. Since c, d are coprime, there exist $a, b \in \mathcal{O}$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. Now the matrix

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^\iota \begin{pmatrix} \delta & \beta \\ 0 & \eta/\delta \end{pmatrix}^\iota$$

belongs to $Mat_2(\mathcal{O})_\eta$. So there exists a matrix $\begin{pmatrix} \delta' & \beta' \\ 0 & \eta/\delta' \end{pmatrix}$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^\iota \begin{pmatrix} \delta & \beta \\ 0 & \eta/\delta \end{pmatrix}^\iota \in \begin{pmatrix} \delta' & \beta' \\ 0 & \eta/\delta' \end{pmatrix} G$. Therefore $\begin{pmatrix} \delta' & \beta' \\ 0 & \eta/\delta' \end{pmatrix}^{-1} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)^\iota \begin{pmatrix} \delta & \beta \\ 0 & \eta/\delta \end{pmatrix}^\iota \in G$. Now we apply the Shimura involution to conclude. □

As for the Hecke operators on Manin symbols for $\Gamma_0(\mathfrak{n})$ we have the following.

Proposition 3.3.10. *Consider the Manin symbol of level $\Gamma_0(\mathfrak{n})$ and weight V :*

$$(0,\cdots,0,\overbrace{(c:d)\otimes P}^{i\text{-}th\ entry},0,\cdots,0)\in\oplus_{i=1}^{r}R[\mathbb{P}^1(\mathcal{O}/\mathfrak{n})]\otimes V.$$

Let $a,b\in\mathcal{O}$ such that $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in G$ and $M=\left(\begin{smallmatrix}\delta&\beta\\0&\eta/\delta\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\left(\begin{smallmatrix}\eta/\delta'&-\beta'\\0&\delta'\end{smallmatrix}\right)^{-1}\in G$ be given by Lemma 3.3.9. Let $\left(\begin{smallmatrix}\eta/\delta'&-\beta'\\0&\delta'\end{smallmatrix}\right)(\{\sigma_i.\infty,\infty\}\otimes P)=\sum_{j=1}^{r}\sum_{k=1}^{s_j}M_{i,j,k}(\{\sigma_j.\infty,\infty\}\otimes M_{i,j,k}^{-1}\left(\begin{smallmatrix}\eta/\delta'&-\beta'\\0&\delta'\end{smallmatrix}\right).P)$. Then the j-th entry of the action of the matrix $\alpha=\left(\begin{smallmatrix}\delta&\beta\\0&\eta/\delta\end{smallmatrix}\right)$ in the set R_η defining the Hecke operator T_η on $(0,\cdots,0,(c:d)\otimes P,0,\cdots,0)$ is given as follows

$$(\alpha.(0,\cdots,0,\overbrace{(c:d)\otimes P}^{i\text{-}th\ entry},0,\cdots,0))_j=\sum_{k=1}^{s_j}(c:d)\left(\begin{smallmatrix}\delta'&-\beta'\\0&\eta/\delta'\end{smallmatrix}\right)M_{i,j,k}\otimes(\left(\begin{smallmatrix}\delta'&-\beta'\\0&\eta/\delta'\end{smallmatrix}\right)M_{i,j,k})^{\iota}.P.$$

Proof. We start from the expansion

$$\sum_{j=1}^{r}\sum_{k=1}^{s_j}M_{i,j,k}(\{\sigma_j.\infty,\infty\}\otimes M_{i,j,k}^{-1}\left(\begin{smallmatrix}\eta/\delta'&-\beta'\\0&\delta'\end{smallmatrix}\right).P)=\left(\begin{smallmatrix}\eta/\delta'&-\beta'\\0&\delta'\end{smallmatrix}\right).(\{\sigma_i\infty,\infty\}\otimes P).$$

Then applying M to the above equality leads to the hybrid equality

$$\alpha.(0,\cdots,0,(c:d)\otimes P,0,\cdots,0)=\sum_{j=1}^{r}\sum_{k=1}^{s_j}MM_{i,j,k}(\{\sigma_j.\infty,\infty\}\otimes M_{i,j,k}^{-1}\left(\begin{smallmatrix}\eta/\delta'&-\beta'\\0&\delta'\end{smallmatrix}\right).P).$$

Now translating the right hand side of this equality into Manin symbols by using the correspondence between Manin symbols and modular symbols, we obtain that the j-entry of the left hand side is

$$\begin{aligned}(\alpha.(0,\cdots,0,(c:d)\otimes P,0,\cdots,0))_j &= \sum_{k=1}^{s_j}(c:d)\left(\begin{smallmatrix}\delta'/\eta&\beta'/\eta\\0&1/\delta'\end{smallmatrix}\right)M_{i,j,k}\otimes M_{i,j,k}^{-1}\left(\begin{smallmatrix}\eta/\delta'&-\beta'\\0&\delta'\end{smallmatrix}\right).P\\ &= \sum_{k=1}^{s_j}(c:d)\left(\begin{smallmatrix}\delta'&\beta'\\0&\eta/\delta'\end{smallmatrix}\right)M_{i,j,k}\otimes(\left(\begin{smallmatrix}\delta'&\beta'\\0&\eta/\delta'\end{smallmatrix}\right)M_{i,j,k})^{\iota}.P.\end{aligned}$$

The second equality is because we have

$$(c:d)\left(\begin{smallmatrix}\delta'/\eta&\beta'/\eta\\0&1/\delta'\end{smallmatrix}\right)=(c\delta'/\eta:c\beta'/\eta+d/\delta')=(\delta'c:\beta'c+d\eta/\delta')=(c:d)\left(\begin{smallmatrix}\delta'&\beta'\\0&\eta/\delta'\end{smallmatrix}\right).$$

This ends the proof of the proposition. $\square$

Remark 3.3.11. *The reader observes immediately that the description of Hecke operators in Proposition 3.3.10 is similar to the one in Proposition 3.3.5. This suggests that the matrices* $\begin{pmatrix} \delta' & \beta' \\ 0 & \eta/\delta' \end{pmatrix} M_{i,j,k}$ *satisfy condition* C_η*, which is indeed the case, see Chapter 4 for details. This also tells us how to look for matrices satisfying condition* C_η*.*

Often, one is interested in computing the Hecke operator T_π for π a prime element of $\mathcal{O}$ coprime with $\mathfrak{n}$. In this instance, we have a more effective description as follows. Let $\mathfrak{p}$ be the prime ideal generated by π.

Proposition 3.3.12. *Let* $(0, \cdots, 0, \overbrace{(c:d) \otimes P}^{i\text{-th entry}}, 0, \cdots, 0) \in \oplus_{i=1}^{r} R[\mathbb{P}^1(\mathcal{O}/\mathfrak{n})] \otimes V$ *be a Manin symbol of level* $\Gamma_0(\mathfrak{n})$ *and weight* V*. Let* α *from a set of representatives of* $\mathcal{O}/\mathfrak{p}$*. Let the modular symbols* $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & \pi \end{smallmatrix}\right).\{\sigma_i.\infty, \infty\} = \{\frac{\sigma_i.\infty+\alpha}{\pi}, \infty\}, \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right).\{\sigma_i.\infty, \infty\}$ *be given as* $\sum_{j=1}^{r}\sum_{k=1}^{s_j} M_{i,j,k}\{\sigma_j.\infty, \infty\}$ *and* $\sum_{j=1}^{r}\sum_{k=1}^{s_j} N_{i,j,k}\{\sigma_j.\infty, \infty\}$*, respectively. Then the matrices* $\gamma_1 = \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)$ *and* $\gamma_\alpha = \left(\begin{smallmatrix} 1 & \alpha \\ 0 & \pi \end{smallmatrix}\right)$ *in* R_π *defining the Hecke operator* T_π *act as follows.*

$$(\gamma_1.(0, \cdots, 0, \overbrace{(c:d) \otimes P}^{i\text{-th entry}}, 0, \cdots, 0))_j = \sum_{k=1}^{s_j} (c:d) \left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi \end{smallmatrix}\right) N_{i,j,k} \otimes (\left(\begin{smallmatrix} 1 & 0 \\ 0 & \pi \end{smallmatrix}\right) N_{i,j,k})^\iota.P.$$

$$(\gamma_\alpha.(0, \cdots, 0, (c:d) \otimes P, 0, \cdots, 0))_j = \sum_{k=1}^{s_j} (c:d) \left(\begin{smallmatrix} \pi & -\alpha \\ 0 & 1 \end{smallmatrix}\right) M_{i,j,k} \otimes (\left(\begin{smallmatrix} \pi & -\alpha \\ 0 & 1 \end{smallmatrix}\right) M_{i,j,k})^\iota.P.$$

Proof. For the first statement since $\mathfrak{p}$ is coprime with $\mathfrak{n}$, by the Chinese Remainder Theorem we can assume that c lies in $\mathfrak{p}$ so that $c = \pi c'$. Given $a, b \in \mathcal{O}$ such that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ we have

$$M_1 = \left(\begin{smallmatrix} a & \pi b \\ c' & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \in G.$$

We next write

$$\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right).\{\sigma_i.\infty, \infty\} \otimes \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right).P = \sum_{j=1}^{r}\sum_{k=1}^{s_j} N_{i,j,k}(\{\sigma_j.\infty, \infty\} \otimes N_{i,j,k}^{-1} \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right).P).$$

We apply M_1 to the above equality to obtain the hybrid equality

$$\left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right).(0, \cdots, 0, (c:d) \otimes P, 0, \cdots, 0) = \sum_{j=1}^{r}\sum_{k=1}^{s_j} M_1 N_{i,j,k}(\{\sigma_j.\infty, \infty\} \otimes N_{i,j,k}^{-1} \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right).P).$$

Now by the correspondence between Manin symbols and modular symbols, one obtains that the left hand side has j-entry given as

$$\left(\left(\begin{smallmatrix}\pi & 0\\ 0 & 1\end{smallmatrix}\right).(0,\cdots,0,(c:d)\otimes P,0,\cdots,0)\right)_j = \sum_{k=1}^{s_j}(c:d)\left(\begin{smallmatrix}1 & 0\\ 0 & \pi\end{smallmatrix}\right)N_{i,j,k}\otimes\left(\left(\begin{smallmatrix}1 & 0\\ 0 & \pi\end{smallmatrix}\right)N_{i,j,k}\right)^{\iota}.P.$$

This ends the proof of the first statement.

As for the second statement, the main observation is that we can choose $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in G$ corresponding to $(c:d)$ such that

$$M = \left(\begin{smallmatrix}1 & \alpha\\ 0 & \pi\end{smallmatrix}\right) g \left(\begin{smallmatrix}1 & \alpha\\ 0 & \pi\end{smallmatrix}\right)^{-1} \in G.$$

Indeed, following Cremona [14], we can assume that $c\alpha \not\equiv d \pmod{\pi}$ after replacing d by $d+\alpha$ where $\alpha \in \mathfrak{n}$ if necessary. For any $a, b \in \mathcal{O}$ such that $ad - bc = 1$, solve for x the congruence

$$(c\alpha - d)x \equiv (b + d\alpha) - \alpha(a + c\alpha) \pmod{\pi}.$$

Now we have $(a + cx)d - (b + dx)c = 1$ and also

$$(b + dx + d\alpha) - r(a + cx + c\alpha) = \pi b'$$

where $b' \in \mathcal{O}$. Next one verifies that the matrix

$$M = \begin{pmatrix}a+cx+\alpha c & b'\\ \pi c & d-\alpha c\end{pmatrix} = \left(\begin{smallmatrix}1 & \alpha\\ 0 & \pi\end{smallmatrix}\right)\left(\begin{smallmatrix}a+cx & b+dx\\ c & d\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \alpha\\ 0 & \pi\end{smallmatrix}\right)^{-1} \in G$$

as needed. We end the proof similarly as in the proof of the first statement. □

So far, following Merel's approach we have firstly described Hecke operators under the existence of families of matrices satisfying what we have called Merel's condition C_η or C_Δ. Secondly in what we call Cremona's approach to Hecke operators on Manin symbols, one needs to have in hand expansions of modular symbols into classical Manin symbols. The next chapter is devoted to the construction of families of matrices achieving these assumptions.

Chapter 4

Heilbronn-Merel Families

From the euclidean algorithm Heilbronn-Merel matrices describing Hecke operators on Manin symbols are constructed. A statement about the L-series associated to an eigenform is derived from the theory described in Chapter 3 with the comparison of modular symbols and group cohomology. Some experimental data will be given.

4.1 Introduction

Let F be an imaginary quadratic field of class number one with $\mathcal{O}$ as its ring of integers. As in Chapter 3, G is $\mathrm{SL}_2(\mathcal{O})$. We keep the set of generators of G as in Chapter 3. We will construct families of matrices satisfying condition C_η. One of these families is reminiscent of Heilbronn matrices . These are matrices with positive integer entries and can be used to define Hecke operators on Manin symbols as shown by Merel [30]. In the classical setting, a Heilbronn matrix of determinant n is a matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ satisfying $a > b \geq 0$ and $0 \leq c < d$. This type of matrices has been used by Heilbronn [24] in order to derive an asymptotic formula for the length of a class of finite continued fractions. Their first usage in the theory of modular symbols is due to Manin. In [30], Merel showed that the set

$$\chi_{1,n} = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Mat_2(\mathbb{Z}) : a > b \geq 0, 0 \leq c < d, ad - bc = n \right\}$$

satisfies condition C_n and hence describes the Hecke operator T_n on Manin symbols. Lastly in [14], Cremona has shown that by the euclidean algorithm one can construct a slight variant of the set $\chi_{1,n}$ describing the Hecke operator T_n on Manin symbols. Denote the Cremona set of matrices defining T_n as $\chi'_{1,n}$, then it turns out that $\chi_{1,n}$ and $\chi'_{1,n}$ are essentially the same. By this we mean that $\chi_{1,n}$ can be constructed from a continued fraction algorithm as $\chi'_{1,n}$.

In the euclidean imaginary quadratic fields, we shall next construct a set of matrices

$$\chi_{1,\eta} \subset \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in Mat_2(\mathcal{O}) : N(a) > N(b) \geq 0, 0 \leq N(c) < N(d), ad - bc = \eta \right\}$$

which describes the Hecke operator T_η on Manin symbols. In the non-euclidean case, a set of matrices defining T_η will be given under the assumption that a word decomposition of an element $g \in G$ is available.

As by-product of the theory described in Chapter 3, we obtain a statement concerning L-series associated with eigenforms over imaginary quadratic fields. See Proposition 4.4.4 for the precise statement. This is done in Section 4.4 where we compare modular symbols and group cohomology among other things.

Aside from the modular symbol formalism for computing modular forms over imaginary quadratic fields, there are others approaches to the computation of modular forms. For instance given a presentation of G, then one can compute directly the cohomology of G with the help of a computer. This is the method used by Haluk Şengün to compute H^1 of G, see [28] for details. The Heilbronn-Merel matrices that we shall define in the sequel describe Hecke operators on cohomology. The main observation that one can make based upon computations performed by Şengün is that the computation of Hecke operators on modular forms via Heilbronn-Merel matrices is faster. But also, the computation of modular forms via the modular symbols formalism is more efficient against a direct computation. See Section 4.5 where we report about these computations.

4.2 The euclidean case

We recall the notation we shall use. Let $0 \neq \eta \in \mathcal{O}$ and $\mathcal{D}$ a set of "positive" divisors of η as defined before Lemma 3.3.6. Until further notice, we are working with the imaginary quadratic fields $F = \mathbb{Q}(\sqrt{-d})$, where $d \in \{1, 2, 3, 7, 11\}$. We set

$$\varepsilon = \begin{cases} (\frac{1+d}{4\sqrt{d}})^2 & \text{if } d \equiv 3 \pmod 4 \\ \frac{1+d}{4} & \text{otherwise.} \end{cases}$$

The euclidean distance from an algebraic number in one of those named field to its nearest algebraic integers is at most $\sqrt{\epsilon}$. See [12] for details.

For $\delta \in \mathcal{D}$ choose a set of representatives S_δ of $\mathcal{O}/\delta\mathcal{O}$ such that for each $\beta \in S_\delta$ we have $N(\beta) < N(\delta)$. Such a set of representatives always exists. Indeed given a set of representatives for $\mathcal{O}/\delta\mathcal{O}$, say A_δ, then for $\alpha \in A_\delta$ let α' be a remainder of the division of α by δ. The collection of α' for $\alpha \in A_\delta$ is a set of representatives of S_δ which has the desired property.

In what follows the quotient of a division is a nearest integral element. As we know this will provide us with an euclidean algorithm in F, the euclidean map being the norm. We also know that in case of ambiguity in the choice of a nearest integer, any choice will do. To illustrate this suppose that $F = \mathbb{Q}(\sqrt{-d})$ with $d = 1, 2$. The integers ring of F is $\mathcal{O} = \mathbb{Z}[\zeta]$ where $\zeta = \sqrt{-d}$. Let $x, y \in \mathcal{O}$ and write $\frac{x}{y} = \alpha + \beta\zeta$ with $\alpha, \beta \in \mathbb{Q}$. Let $a, b \in \mathbb{Z}$ such that $|\alpha - a| \leq \frac{1}{2}$ and $|\beta - b| \leq \frac{1}{2}$ (a is a nearest integer to α and it is the same for b with regard to β). Define $q = a + b\zeta \in \mathcal{O}$. Then $x = qy + z$ where $z \in \mathcal{O}$ is such that: $N(z) \leq \frac{1+d}{4}N(y) < N(y)$ since $\frac{1+d}{4} < 1$.

The set of Heilbronn-Merel matrices of determinant η is obtained as follows. For $\beta \in S_\delta$ we let $x_0 = \delta, x_1 = \beta, y_0 = 0, y_1 = \eta/\delta$. We form then the matrix $M_1 = \left(\begin{smallmatrix} x_0 & x_1 \\ y_0 & y_1 \end{smallmatrix}\right)$. From this we form the second matrix $M_2 = \left(\begin{smallmatrix} x_1 & x_2 \\ y_1 & y_2 \end{smallmatrix}\right)$ where x_2 and y_2 are defined as follows:

1. $x_2 = x_1 q_1 - x_0$, with $-x_2$ a remainder obtained from the division of x_0 by x_1.

2. $y_2 = y_1 q_1 - y_0$.

Both M_1, M_2 have determinant η. We have $N(x_2) \leq \varepsilon N(x_1)$. We also have by definition of the matrices M_1, M_2, that $M_1.\infty := \infty, M_1.0 = M_2.\infty$. So, generally from a matrix $M_i = \left(\begin{smallmatrix} x_{i-1} & x_i = x_{i-1}q_{i-1} - x_{i-2} \\ y_{i-1} & y_i = y_{i-1}q_{i-1} - y_{i-2} \end{smallmatrix}\right)$ we form the matrix $M_{i+1} = \left(\begin{smallmatrix} x_i & x_{i+1} = x_i q_i - x_{i-1} \\ y_i & y_{i+1} = y_i q_i - y_{i-1} \end{smallmatrix}\right)$. We stop the process once the remainder is zero. Then by definition we have $N(x_i) \leq \varepsilon N(x_{i-1}) \leq \cdots \leq \varepsilon^{i-1}N(x_1)$, and $det(M_i) = \eta, M_i.0 = M_{i+1}.\infty$. If say M_s is the last matrix, we have $M_s.0 = 0$. It is worth noting that M_i and M_{i+1} are related by the equality

$$M_{i+1} = M_i \left(\begin{smallmatrix} 0 & -1 \\ 1 & q_i \end{smallmatrix}\right).$$

The other property which is satisfied by the bottom rows of the matrices M_i reads as follows.

Proposition 4.2.1 (Hurwitz [25], Poitou [31]). *For each i the inequality $N(y_i) < N(y_{i+1})$ or equivalently $|y_i| < |y_{i+1}|$ holds.*

Hurwitz's result deals with approximation of complex numbers by Gaussian integers. He uses continued fractions to do so. Poitou's result is concerned with the same problem of approximating complex numbers but in addition to the Gaussian integers he considers the other quadratic imaginary fields endowed with an euclidean division. The tools are continued fractions as well. They both showed that the denominators of the convergents appearing increase in absolute value.

We shall next see that the bottom rows of the M_i are precisely the denominators of the convergents of the continued fraction of $\alpha = x_1/x_0$ up to multiplication by $-\eta/\delta$. By construction the bottom row of M_i is (y_{i-1}, y_i) where $y_i = q_{i-1}y_{i-1} + (-y_{i-2})$ with $y_0 = 0$ and $y_1 = \eta/\delta$. On the other hand we have

$$\begin{aligned}
x_0 &= q_1x_1 + (-x_2) \Leftrightarrow \frac{x_1}{x_0} = \frac{1}{q_1 + \frac{-x_2}{x_1}} \\
x_1 &= (-q_2)(-x_2) + (-x_3) \Leftrightarrow \frac{-x_2}{x_1} = \frac{1}{-q_2 + \frac{x_3}{x_2}} \\
&\vdots \\
x_n &= \begin{cases} q_{n+1}x_{n+1} + (-x_{n+2}) \Leftrightarrow \frac{x_{n+1}}{x_n} = \frac{1}{q_{n+1}+\frac{-x_{n+2}}{x_{n+1}}} & \text{if } n \text{ is even} \\ (-q_{n+1})(-x_{n+1}) + (-x_{n+2}) \Leftrightarrow \frac{-x_{n+1}}{x_n} = \frac{1}{-q_{n+1}+\frac{x_{n+2}}{x_{n+1}}} & \text{if } n \text{ is odd.} \end{cases}
\end{aligned}$$

Hence a continued fraction of α is given as

$$\frac{x_1}{x_0} = [q_0; q_1, -q_2, q_3, \cdots, (-1)^{n+2}q_{n+1}]$$

where $q_0 = 0$. Here, $[q_0; q_1, -q_2]$ is the shorthand for $q_0 + \frac{1}{q_1+\frac{1}{-q_2}}$. Now the n-th convergent to α which we denote by c_n is defined as

$$c_n = [q_0; q_1, -q_2, q_3, \cdots, (-1)^{n+1}q_n] = [q_0; k_1, k_2, k_3, \cdots, k_n]$$

where $k_i = (-1)^{i+1}q_i$. It can be shown that $c_n = \frac{a_n}{b_n}$ with a_n and b_n coprime such that:

$$a_n = k_na_{n-1} + a_{n-2};\ b_n = k_nb_{n-1} + b_{n-2}$$

with $a_{-1} = 1, a_0 = 0, b_{-1} = 0$ and $b_0 = 1$. For $n = 1, 2$, we have

$$c_1 = \frac{a_1}{b_1} = [q_0; q_1] = \frac{1}{q_1} \text{ and } c_2 = \frac{a_2}{b_2} = [q_0; q_1, -q_2] = \frac{1}{q_1 + \frac{1}{-q_2}} = \frac{-q_2}{-q_2q_1 + 1}.$$

These agree with the recursive formulas for $n = 1, 2$. Now suppose that the formulas are true up to n and let us check that the formulas hold for $n + 1$. We have $c_{n+1} = [k_0; k_1, \cdots, k_{n+1}] = [k_0; k_1, \cdots, k'_n]$ with $k'_n = k_n + \frac{1}{k_{n+1}}$. By induction hypothesis we have

$$\begin{aligned} c_{n+1} &= \frac{k'_n a_{n-1} + a_{n-2}}{k'_n b_{n-1} + b_{n-2}} \\ &= \frac{k_{n+1}k_n a_{n-1} + k_{n+1}a_{n-2} + a_{n-1}}{k_{n+1}k_n b_{n-1} + k_{n+1}b_{n-2} + b_{n-1}} \\ &= \frac{k_{n+1}(k_n a_{n-1} + a_{n-2}) + a_{n-1}}{k_{n+1}(k_n b_{n-1} + b_{n-2}) + b_{n-1}} \\ &= \frac{k_{n+1}a_n + a_{n-1}}{k_{n+1}b_n + b_{n-1}} = \frac{a_{n+1}}{b_{n+1}}. \end{aligned}$$

For $n \geq 1$, we let $b'_n = \eta/\delta b_{n-1}$.

Lemma 4.2.2. *For $n \geq 1$, we have that $b'_n = \pm y_n$.*

Proof. We first write the first few terms in the sequences y_n and b_n. We have

$$\begin{aligned} b'_1 = \eta/\delta \quad &\text{and} \quad y_1 = \eta/\delta \\ b'_2 = \eta/\delta(k_1 b_0 + b_{-1}) = \eta/\delta q_1 \quad &\text{and} \quad y_2 = q_1 y_1 - y_0 = \eta/\delta q_1 \\ b'_3 = \eta/\delta(k_2 b_1 + b_0) = \eta/\delta(-q_2 q_1 + 1) \quad &\text{and} \quad y_3 = q_2 y_2 - y_1 = \eta/\delta(q_2 q_1 - 1). \end{aligned}$$

So suppose that $b'_{2n-1} = y_{2n-1}$ and $b'_{2n} = y_{2n}$. Then $b'_{2n+1} = b_{2n} = k_{2n}b_{2n-1} + b_{2n-2} = k_{2n}b'_{2n} + b'_{2n-1} = -q_{2n}y_{2n} + y_{2n-1} = -y_{2n+1}$ and $b'_{2n+2} = b_{2n+1} = k_{2n+1}b_{2n} + b_{2n-1} = k_{2n+1}b'_{2n+1} + b'_{2n} = -q_{2n+1}y_{2n+1} + y_{2n} = -y_{2n+2}$. Similarly if we had $b'_{2n-1} = -y_{2n-1}$ and $b'_{2n} = -y_{2n}$, one gets $b'_{2n+1} = y_{2n+1}$ and $b'_{2n+2} = y_{2n+2}$. $\square$

This lemma, the results of Hurwitz and Poitou show that the set of matrices we have constructed, namely be setting $M_1 = \begin{pmatrix} \delta & \beta \\ 0 & \eta/\delta \end{pmatrix}$, then

$$\chi_{1,\eta} = \bigcup_{\delta \in \mathcal{D}} \bigcup_{\beta \in S_\delta} \left\{ M_i := \begin{pmatrix} x_{i-1} & x_i \\ y_{i-1} & y_i \end{pmatrix} : M_{i+1} = M_i \begin{pmatrix} 0 & -1 \\ 1 & q_i \end{pmatrix}, \text{ and } x_{i+1} = x_i q_i - x_{i-1} \right\}$$

has all its elements $\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfying $N(a) > N(b) \geq 0$ and $0 \leq N(c) < N(d)$. Therefore any element in $\chi_{1,\eta}$ has entries with norms less or equal to the norm of η.

By construction the element $\Sigma_1 = \sum_{\theta \in \chi_{1,\eta}} \theta \in R[Mat_2(\mathcal{O})_\eta]$ satisfies condition C_η, which in this situation takes the form: for $K \in Mat_2(\mathcal{O})_\eta/G$ we have

$$\sum_{\theta \in K} \theta\{0, \infty\} = \{0, \infty\}.$$

Indeed from the relation $M_{i+1} = M_i \begin{pmatrix} 0 & -1 \\ 1 & q_i \end{pmatrix}$, we deduce that for all i, M_i is in the class of $M_1 = \begin{pmatrix} \delta & \beta \\ 0 & \eta/\delta \end{pmatrix}$. From the properties $M_1.\infty = \infty, M_s.0 = 0$ and $M_i.0 = M_{i+1}.\infty$, one has that $\sum_{\theta \in K} \theta\{\infty, 0\} = \{\infty, 0\}$. This is equivalent to the claimed equality.

Here is the pseudo-algorithm for computing the Heilbronn-Merel matrices in the euclidean case. It is inspired from an algorithm given in [14].

Algorithm 4.2.3. Input: *A non-zero element $\eta \in \mathcal{O}$.*
Output: *A list of matrices $\chi_{1,\eta}$ satisfying condition C_η.*

1. *Form $L = [\ ]$ and $M_0 = \begin{pmatrix} 1 & 0 \\ 0 & \eta \end{pmatrix}$.*

2. *for $\delta \in \mathcal{D} - \{1\}$ do*

   (a) *for $\beta \in S_\delta$ do*

   $x_0 = \delta; x_1 = \beta; y_0 = 0; y_1 = \eta/\delta;\ M_1 = \begin{pmatrix} x_0 & x_1 \\ y_0 & y_1 \end{pmatrix};$
   append M_1 to L;
   while $x_1 \neq 0$ do
   $q = x_0\ div\ x_1$; *//the quotient of the division of x_0 by x_1*
   $x_2 = qx_1 - x_0; x_0 = x_1; x_1 = x_2;$
   $y_2 = qy_1 - y_0; y_0 = y_1; y_1 = y_2;$
   $M_2 = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}$;
   append M_2 to L;
   end while;

   (b) *end for;*

3. *end for;*

4. *append M_0 to L;*

5. *return L;*

In the above algorithm one could take another set of representatives S_δ of $\mathcal{O}/\delta\mathcal{O}$ which does not satisfy the norm condition, and the resulting matrices will no longer satisfy the property concerning the norms of the entries. In this case, the number of Heilbronn-Merel matrices produced by the above algorithm increases and the norms of the entries can be greater than the norm of the input. But, of course they will satisfy condition C_η.

Remark 4.2.4. *Because for non-zero and coprime $\beta, \eta \in \mathcal{O}$, we have $T_{\beta\eta} = T_\beta T_\eta = T_\eta T_\beta$, we need only to implement Algorithm 4.2.3 for powers of prime elements.*

We next give the complete outputs of the above algorithm in the setting $F = \mathbb{Q}(\sqrt{-2})$ and $\eta = 1+w, (1+w)^2$ respectively, where $w = \sqrt{-2}$. Then Algorithm 4.2.3 implemented in MAGMA [8] gives the following set describing the Hecke operator T_{1+w} :

$$\left\{ \left(\begin{smallmatrix} w+1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} w+1 & -1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 1 & -1-w \end{smallmatrix}\right), \left(\begin{smallmatrix} w+1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1+w \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1+w \end{smallmatrix}\right) \right\}.$$

The Heilbronn-Merel matrices of determinant $(1+w)^2$ describing the Hecke operator $T_{(1+w)^2}$ are as follows:

$$\begin{aligned} &\left\{ \left(\begin{smallmatrix} w+1 & 0 \\ 0 & w+1 \end{smallmatrix}\right), \left(\begin{smallmatrix} w+1 & -1 \\ 0 & w+1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ w+1 & -2w+1 \end{smallmatrix}\right), \left(\begin{smallmatrix} w+1 & 1 \\ 0 & w+1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ w+1 & 2w-1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2w-1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \right. \\ &\left(\begin{smallmatrix} 2w-1 & w \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} w & 1 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 2w-1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2w-1 & w+1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} w+1 & 0 \\ 1 & w+1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2w-1 & -2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -2 & 1 \\ 1 & -w \end{smallmatrix}\right), \\ &\left(\begin{smallmatrix} 1 & 0 \\ -w & 2w-1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2w-1 & -1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 1 & -2w+1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2w-1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 2w+1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2w-1 & 2 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & -1 \\ 1 & w-1 \end{smallmatrix}\right), \\ &\left(\begin{smallmatrix} -1 & 0 \\ w-1 & -2w+1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2w-1 & -w-1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -w-1 & 0 \\ 1 & -w-1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2w-1 & w \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} w & -1 \\ 1 & -w-2 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ -w-2 & -2w+1 \end{smallmatrix}\right), \\ &\left. \left(\begin{smallmatrix} 1 & 0 \\ 0 & 2w-1 \end{smallmatrix}\right) \right\}. \end{aligned}$$

4.3 The non-euclidean class number one case

If one wants to produce a set of matrices $\chi_\eta \subset Mat_2(\mathcal{O})_\eta$ satisfying condition C_η in the situation of non-euclidean imaginary quadratic fields of class number one, one can proceed as follows. We take as granted the existence of a subroutine that computes a word decomposition of an element $g \in G$. This can be done, however, it is not an easy task in general. From Proposition 3.3.7, we know that for all $\gamma_l \in R_\eta$, we can write $\{\gamma_l^{-1}\sigma_i.\infty, \infty\}$ as

$$\{\gamma_l^{-1}\sigma_i.\infty, \infty\} = \sum_{j=1}^{r} \sum_{k=1}^{s_j} S_{l,i,j,k}\{\sigma_j.\infty, \infty\}$$

with $S_{l,i,j,k} \in G$. Applying the matrix γ_l to the above expansion, we obtain

$$\{\sigma_i.\infty, \infty\} = \sum_{j=1}^{r} \sum_{k=1}^{s_j} \gamma_l S_{l,i,j,k}\{\sigma_j.\infty, \infty\}.$$

Next, we form the set of matrices

$$\chi_{i,\eta} = \{\gamma_l S_{l,i,j,k} : 1 \le j \le r, 1 \le k \le s_j, \gamma_l \in R_\eta\}.$$

Proposition 4.3.1. *The set $\chi_{i,\eta}$ satisfies condition C_η.*

Proof. We need to verify that for all classes $K \in Mat_2(\mathcal{O})_\eta/G$ we have

$$\sum_{j=1}^{r} \sum_{\theta \in K \cap \chi_{i,\eta}} \theta\{\sigma_j.\infty, \infty\} = \{\sigma_i.\infty, \infty\}.$$

Let γ_l be the representative of a class K belonging to R_η, then by construction of $\chi_{i,\eta}$ we can write

$$\sum_{j=1}^{r} \sum_{\theta \in K \cap \chi_{i,\eta}} \theta\{\sigma_j.\infty, \infty\} = \sum_{j=1}^{r} \sum_{k=1}^{s_j} \gamma_l S_{l,i,j,k}\{\sigma_j.\infty, \infty\} = \{\sigma_i.\infty, \infty\}.$$

□

Therefore by taking as black-box the word decomposition of an element $g \in G$, we have a procedure to produce a family of matrices satisfying condition C_η.

4.4 Comparison of Hecke modules and universal L-series

We shall recall the isomorphisms linking modular symbols and (co)homology of homogeneous spaces. In the end we will give an application of the theory of Hecke operators we just described to obtain what we call universal L-series associated to cuspidal eigenforms over imaginary quadratic fields.

4.4.1 Comparison of Hecke modules

Steinberg module and modular symbols of weight two

For most of the assertions we shall make in this subsection, we refer to [36] and the references therein for more details. The Steinberg module we are concerned about here is an avatar of the modular symbols of weight two, $\mathcal{M}_2$ defined in Definition 3.2.2. Here is how the Steinberg module denoted as St is defined. Elements from $\mathcal{O}^2$ are viewed as column vectors. Let R be an $\mathcal{O}$-algebra. The Steinberg module is the R-module on the symbols $[v_1, v_2]$ where $v_i = \left(\begin{smallmatrix} a_i \\ c_i \end{smallmatrix}\right) \in \mathcal{O}^2$ and $gcd(a_i, c_i) = 1$, subject to the relations: $[v_1, v_2] = -[v_2, v_1], [v_1, v_2] + [v_2, v_3] = [v_1, v_3]$ and $[v_1, v_2] = 0$ if $det([v_1, v_2]) = det(\left(\begin{smallmatrix} a_1 & a_2 \\ c_1 & c_2 \end{smallmatrix}\right)) = 0$.

Recall the definition of $\mathcal{M}_2$. It is the R-module on the symbols $\{\alpha, \beta\} : R[\{\alpha, \beta\}]/ < \{\alpha, \alpha\}, \{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\}, \alpha, \beta \in \mathbb{P}^1(F) >$. The homomorphism from St to $\mathcal{M}_2$ mapping $[\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)]$ to $\{a/c, b/d\}$ defines an isomorphism of R-modules: $St \cong \mathcal{M}_2$. This is because of the following observations. First the set $\{\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right) : gcd(a, c) = 1\}$ is the projective line over $\mathcal{O}$. The latter is also $\mathbb{P}^1(F)$. Now the relations defining St are equivalent to the relations defining $\mathcal{M}_2$. The condition $[\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)] = 0$ if $det([\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right), \left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)]) = 0$ is equivalent to the relation $\{\alpha, \alpha\}$, because $\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} b \\ d \end{smallmatrix}\right)$ are then linearly dependent, thus $a/c = b/d$ in $\mathbb{P}^1(F)$. The relation $[v_1, v_2] + [v_2, v_3] + [v_3, v_1] = 0$, which implies the relation $[v_1, v_2] + [v_2, v_1] = 0$ by taking $v_3 = v_1$, is clearly equivalent to the relation $\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0$.

Let Γ be a congruence subgroup of G. We recall from [39] the definition of parabolic cohomology. Take G_∞, the stabilizer of ∞ in G as defined before Proposition 3.2.1. The parabolic cohomology $\mathrm{H}^i_{\mathrm{par}}(\Gamma, V)$ of Γ with coefficients in a Γ-module V is defined by the following exact sequence:

$$0 \to \mathrm{H}^i_{\mathrm{par}}(\Gamma, V) \to \mathrm{H}^i(\Gamma, V) \xrightarrow{res} \prod_{g \in \Gamma \backslash G / G_\infty} \mathrm{H}^i(\Gamma \cap g G_\infty g^{-1}, V).$$

It is worth mentioning that because of the bijection $G/G_\infty \longleftrightarrow \mathbb{P}^1(F)$, the double cosets $\Gamma \backslash G / G_\infty$ are in fact the Γ-cusps: $\Gamma \backslash \mathbb{P}^1(F)$.

Now suppose that the torsion elements in Γ have orders invertible in R. There is the following important theorem which establishes the link between modular symbols and group cohomology of Γ.

Theorem 4.4.1. *Keeping the same assumptions as above, then there are isomorphisms of R-modules*

1. $\mathrm{H}^2(\Gamma, V) \cong \mathcal{M}_R(\Gamma, V)$
2. $\mathrm{H}^2_{par}(\Gamma, V) \cong \mathcal{CM}_R(\Gamma, V)$.

This theorem is deduced from Borel-Serre duality which provides in fact an isomorphism $\mathrm{H}^2(\Gamma, V) \cong \mathrm{H}_0(\Gamma, St \otimes_R V) = (\mathcal{M}_2 \otimes_R V)_\Gamma = \mathcal{M}_R(\Gamma, V)$, see [1] or [36, p. 61] for more details. The second isomorphism follows from the first and the definition of parabolic cohomology. Furthermore, one can define Hecke operators on $\mathrm{H}^2(\Gamma, V)$ and the above isomorphisms respect the Hecke action on both sides.

4.4.2 Eichler-Shimura-Harder isomorphism

At this point we are following Taylor [34, chap. 4]. Let $\overline{\Gamma \backslash \mathbb{H}_3}$ be the Borel-Serre compactification of $\Gamma \backslash \mathbb{H}_3$. This is a compact manifold with boundary and a homotopy equivalence $i : \Gamma \backslash \mathbb{H}_3 \hookrightarrow \overline{\Gamma \backslash \mathbb{H}_3}$. Consider the locally constant sheaf $\mathcal{V}_{r,s}(\mathbb{C})$ on $\Gamma \backslash \mathbb{H}_3$ associated with $V_{r,s}(\mathbb{C})$. Via i, the sheaf $\mathcal{V}_{r,s}(\mathbb{C})$ extends to a sheaf $i_*(\mathcal{V}_{r,s}(\mathbb{C}))$ on $\overline{\Gamma \backslash \mathbb{H}_3}$, and one has the identification

$$\mathrm{H}^i(\overline{\Gamma\backslash\mathbb{H}_3}, i_*(\mathcal{V}_{r,s}(\mathbb{C}))) \cong \mathrm{H}^i(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,s}(\mathbb{C})).$$

The cuspidal $\mathrm{H}^i_{cusp}(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,s}(\mathbb{C}))$ and Eisenstein $\mathrm{H}^i_{Eis}((\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,s}(\mathbb{C}))$ cohomologies are defined as the kernel and the image of the following homomorphism respectively:

$$\mathrm{H}^i(\overline{\Gamma\backslash\mathbb{H}_3}, \mathcal{V}_{r,s}(\mathbb{C})) \to \mathrm{H}^i(\partial\overline{\Gamma\backslash\mathbb{H}_3}, \mathcal{V}_{r,s}(\mathbb{C})).$$

Let $\mathbb{A}_f$ be the finite adeles of F and let $U(\Gamma)$ be the closure of Γ in $\mathrm{SL}_2(\mathbb{A}_f)$, that is $U(\Gamma)$ is such that $\Gamma = U(\Gamma) \cap \mathrm{SL}_2(K)$. Let $S_n(\Gamma, \mathbb{C}) := \oplus\rho_f^{U(\Gamma)}$, where the sum is over all the cuspidal automorphic representations $\rho = \rho_f \otimes \rho_\infty$ of SL_2 over F with ρ_∞ the principal series representation of $\mathrm{SL}_2(\mathbb{C})$ induced by the character: $\left(\begin{smallmatrix} a & * \\ 0 & a^{-1} \end{smallmatrix}\right) \mapsto (\frac{a}{|a|})^{2n+2}$. We collect some of the theorems obtained by Günter Harder.

Theorem 4.4.2 (Harder). *1. For all r, s and $i > 2$, we have $\mathrm{H}^i_{cusp}(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,s}(\mathbb{C})) = 0$. For $1 \leq i \leq 2$, $\mathrm{H}^i_{cusp}(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,s}(\mathbb{C})) = 0$ unless $r = s$.*

2. $\mathrm{H}^1_{cusp}(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,r}(\mathbb{C})) \cong \mathrm{H}^2_{cusp}(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,r}(\mathbb{C})) \cong S_r(\Gamma, \mathbb{C})$.

3. $\mathrm{H}^0_{Eis}(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,s}(\mathbb{C})) = 0$ unless $r = s$ in which case it is $\mathbb{C}$.

Following [34], we get the description in terms of group cohomology as follows. One defines Γ-cusps to be a Γ-conjugacy class of Borel subgroups of $\mathrm{SL}_2(F)$. Set $\Gamma_B = \Gamma \cap B$, for B a Borel of $\mathrm{SL}_2(F)$. One also defines the following abelian group

$$\mathrm{H}^i_\partial(\Gamma, V_{r,s}(\mathbb{C})) = \oplus_B \mathrm{H}^i(\Gamma_B, V_{r,s}(\mathbb{C})).$$

Then there is a commutative diagram

$$\begin{array}{ccc} \mathrm{H}^i(\Gamma, V_{r,s}(\mathbb{C})) & \xrightarrow{res} & \mathrm{H}^i_\partial(\Gamma, V_{r,s}(\mathbb{C})) \\ \downarrow & & \downarrow \\ \mathrm{H}^i(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{r,s}(\mathbb{C})) & \longrightarrow & \mathrm{H}^i(\partial\overline{\Gamma\backslash\mathbb{H}_3}, \mathcal{V}_{r,s}(\mathbb{C})) \end{array}$$

where the vertical arrows are isomorphisms. So the cuspidal cohomology defined above is just the parabolic cohomology $\mathrm{H}^i_{par}(\Gamma, V_{r,s}(\mathbb{C}))$ as defined above since we have a bijection:

$$\Gamma\backslash G/G_\infty \longleftrightarrow \{\Gamma\text{-}cusps\}; \gamma \mapsto \gamma G_\infty \gamma^{-1},$$

see [34] for details.

4.4.3 Universal L-series

As above we are taking $\mathbb{C}$-coefficients. We consider the congruence subgroup of level one $G = \mathrm{SL}_2(\mathcal{O})$. The imaginary quadratic field F we are dealing with in this subsection is one of the five euclidean imaginary quadratic fields. Because of Theorem 4.4.2, we can view the space of cuspidal modular forms over F of level G and weight $V_{r,s}(\mathbb{C})$ as the cohomology group $\mathrm{H}^1_{par}(G, V_{r,s}(\mathbb{C}))$. We denote the latter by H.

The Hecke operators T_η defined by the sets χ_η of Heilbronn-Merel matrices defined in page 77 also act on the space H. Let $\chi = \cup_{\eta\in\mathcal{O}}\chi_\eta$ where the union is taken up to units. Let $u \in \mathcal{O}^*$ be a unit and consider $u\eta$. The Hecke operators $T_{u\eta}$ and T_η verify the relation:

$$T_{u\eta} = \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right) T_\eta.$$

Let $J = \left(\begin{smallmatrix} u & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In the case where the group of units $\mathcal{O}^*$ has order 2, then J induces an involution on the space H, and so splits the latter into two J-eigenspaces. Indeed, u has order 2, so $J^2 = 1$. In the instance where $\mathcal{O}^*$ has order greater than two, say it is n, then the Hecke operator corresponding to J splits H into n J-eigenspaces. In all cases there is a J-eigenspace with eigenvalue 1. We denote it as H^+.

Let $\mathfrak{a}$ be a non-zero ideal of $\mathcal{O}$. We define $\chi_\mathfrak{a}$ as χ_η for some generator η of $\mathfrak{a}$. Because J acts trivially on H^+, the Hecke operators T_η and $T_{u\eta}$ coincide on H^+. So we can set $T_\mathfrak{a} = T_\eta$ and the set $\chi = \cup_{\mathfrak{a}\subset\mathcal{O}}\chi_\mathfrak{a}$ collects all the Hecke operators on H^+. If $f \in H^+$ is an eigenform for all the Hecke operators $T_\mathfrak{a}$ where $\mathfrak{a} = \eta\mathcal{O}$ with eigenvalue $a_\mathfrak{a}$ at $T_\mathfrak{a}$, then the Hecke L-series $L_f(s)$ associated with f is defined as

$$L_f(s) = \sum_{\mathfrak{a}\subset\mathcal{O}} \frac{a_\mathfrak{a}}{N(\mathfrak{a})^s}.$$

This L-series converges in some right half space, has an Euler product, admits an analytic continuation and satisfies a functional equation as in the classical case.

Next we consider the $\mathbb{C}$-dual $H^\vee$ of H : $H^\vee = \mathrm{Hom}_\mathbb{C}(H, \mathbb{C})$. For any $f \in H^+$, we define a formal L-series associated with a $\mathbb{C}$-linear form $\mathfrak{L} \in H^\vee$ and f as follows.

Definition 4.4.3. *We define the formal L-series $L_{\mathfrak{L},f}(s)$ associated with f and $\mathfrak{L}$ when $\mathfrak{L}(f) \neq 0$ as*

$$L_{\mathfrak{L},f}(s) := \sum_{(\eta)\subset\mathcal{O}} \frac{\mathfrak{L}(T_\eta.f)}{N((\eta))^s} = \sum_{\mathfrak{a}} \frac{\mathfrak{L}(T_\mathfrak{a}.f)}{N(\mathfrak{a})^s}.$$

We have the following proposition.

Proposition 4.4.4. *Let $f \in H^+$ be an eigenform for all the Hecke operators $T_{\mathfrak{a}}$. Let $\mathfrak{L} \in (H^+)^\vee$, the $\mathbb{C}$-dual of H^+ and suppose that $\mathfrak{L}(f) \neq 0$. Then the formal L-series associated with $\mathfrak{L}$ and f defined by*

$$L_{\mathfrak{L},f}(s) = \sum_{M \in \chi} \frac{\mathfrak{L}(M.f)}{N(det(M))^s}$$

is up to a factor the L-series associated with the cuspidal eigenform f for the congruence subgroup G. The constant factor being $\mathfrak{L}(f)$.

Proof. Formally from the formula of Hecke operator $T_{\mathfrak{a}}$ we have

$$L_{\mathfrak{L},f}(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})} \sum_{M \in \chi_{\mathfrak{a}}} \mathfrak{L}(M.f) = \sum_{M \in \chi} \frac{\mathfrak{L}(M.f)}{N(det(M))^s}.$$

If f is such that $T_{\mathfrak{a}}.f = a_{\mathfrak{a}} f$, then the right hand side of the equality in Definition 4.4.3 becomes:

$$\sum_{\mathfrak{a}} \frac{\mathfrak{L}(T_{\mathfrak{a}}.f)}{N(\mathfrak{a})^s} = \mathfrak{L}(f) \sum_{\mathfrak{a} \in \mathcal{O}} \frac{a_{\mathfrak{a}}}{N(\mathfrak{a})^s}.$$

From Theorem 3.3.2 or Proposition 3.3.10, the Hecke operators $T_{\mathfrak{a}}$ on H^+ are described as:

$$T_{\mathfrak{a}}.f = \sum_{M \in \chi_{\mathfrak{a}}} M.f.$$

Thus we have that

$$\sum_{M \in \chi} \frac{\mathfrak{L}(M.f)}{N(det(M))^s} = \mathfrak{L}(f) \sum_{\mathfrak{a} \subset \mathcal{O}} \frac{a_{\mathfrak{a}}}{N(\mathfrak{a})^s}.$$

We deduce then that the formal L-series $L_{\mathcal{L},f}(s)$ is up to a factor the L-series associated with the eigenform f. This ends the proof of the proposition. □

4.5 Experimental data

The experimental computations we shall report here were performed by Haluk Şengün. The computations were done on a 16 Core Intel Xeon machine with 128 GB of ram memory at the Institut für Experimentelle Mathematik.

For $F = \mathbb{Q}(\sqrt{-2})$ and $\mathcal{O} = \mathbb{Z}[w]$ its ring of integers. Şengün has computed Hecke operators on $\mathrm{H}^2(\Gamma_0(\mathfrak{n}), \mathbb{C})$ with $\mathfrak{n}$ an ideal of $\mathcal{O}$ using the Heilbronn-Merel matrices over $\mathcal{O}$ described in previous sections. Over $\mathbb{Q}(\sqrt{-1})$, we will also provide some data comparing direct computations of $\mathrm{H}^1(\Gamma_0(\mathfrak{n}), \mathbb{C})$ with computations of $\mathrm{H}^2(\Gamma_0(\mathfrak{n}), \mathbb{C})$ via modular symbols.

So let $G = \mathrm{SL}_2(\mathcal{O})$. The presentation of G that we shall use here is as follows. Define the matrices:

$$\alpha = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right), \tau = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) \text{ and } \nu = \left(\begin{smallmatrix} 1 & w \\ 0 & 1 \end{smallmatrix}\right).$$

The matrices τ and ν generate G_∞ (the stabiliser of ∞ in G for the linear fractional transformation of G on $\mathbb{P}^1(F)$ defined in Subsection 3.2.1) and we have, see [21] for details

$$G =< \alpha, \tau, \nu : \alpha^4 = (\tau\alpha)^3 = \tau\nu\tau^{-1}\nu^{-1} = (\alpha\nu^{-1}\alpha\nu)^4 = 1 > .$$

Let $\Gamma = \Gamma_0(\mathfrak{n})$ and $V = V_{r,s}(\mathbb{C})$. From Theorem 4.4.1, we know that $\mathrm{H}^2(\Gamma, V)$ is isomorphic to the space of modular symbols of weight V and level Γ over $\mathbb{C}$, $\mathcal{M}_{\mathbb{C}}(\Gamma, V)$. We know from Theorem 3.2.5 that we have the following exact sequence of right $\mathbb{C}[G]$-modules

$$M_{\mathbb{C}}(\Gamma, V) \xrightarrow{g\otimes v \mapsto g\{0,\infty\}\otimes gv} \mathcal{M}_{\mathbb{C}}(\Gamma, V) \to 0$$

where $M_{\mathbb{C}}(\Gamma, V)$ is the space of Manin symbols of weight V and level Γ over $\mathbb{C}$, $\mathbb{C}[\Gamma\backslash G]\otimes_{\mathbb{C}} V$. Let J denote the kernel of this exact sequence so that we have

$$\mathrm{H}^2(\Gamma, V) \cong \mathcal{M}_{\mathbb{C}}(\Gamma, V) \cong M_{\mathbb{C}}(\Gamma, V)/J.$$

The space $M_{\mathbb{C}}(\Gamma, V)$ is computable and so to compute $\mathrm{H}^2(\Gamma, V)$ we need to know J. This J is called the relation ideal of $M_{\mathbb{C}}(\Gamma, V)$. Let M denote the latter. Let $\lambda = \alpha\nu^{-1}\alpha\nu = \left(\begin{smallmatrix} -1 & -w \\ -w & 1 \end{smallmatrix}\right)$. Then the relation ideal is given as follows:

$$J = M(1 + \alpha + \alpha^2 + \alpha^3) + M(1 + \tau\alpha + (\tau\alpha)^2) + M(1 + \lambda - \lambda^2 - \lambda^3).$$

For a proof of this fact see [13] where the relation ideals for all the euclidean imaginary quadratic fields were computed by means of the geometry of each of the five euclidean imaginary quadratic fields. In [36], there is an algebraic computation of the relation ideal for $F = \mathbb{Q}(\sqrt{-1})$. In the non-euclidean class number one setting, in [41] there are computations of the relation ideals by geometric means.

This being said, the theory described in previous sections provides an algorithm for computing system of Hecke eigenvalues on the space $\mathrm{H}^2(\Gamma, V)$ by interpreting the latter as a quotient of Manin symbols $M_{\mathbb{C}}(\Gamma, V)$ and by using the theory of Hecke operators on Manin symbols we described in those sections. For an overview of this modular symbols algorithm see [36] or [13]. Note that the algorithm used in [36] and [13], Hecke operators on Manin symbols are not described. In place, one has to convert Manin symbols into modular symbols in order to compute Hecke operators and convert back to Manin symbols. This is no longer necessary as in the setting of imaginary quadratic fields of class number one, we know now how to describe Hecke operators on Manin symbols intrinsically. It is this algorithm that was used by Şengün to compute the data in Table 4.5.2.

There is also a direct method for computing the cohomology of Γ in degree 1 with trivial or non-trivial coefficients. Let us briefly divagate on that. Because G is finitely presented any value of a 1-cocycle $f : G \to V$ can be written as a linear combination of $f(\alpha), f(\tau), f(\nu)$. Let $Z^1(G, V)$ be the space of 1-cocycle and $D^1(G, V)$ its subspace of coboundaries. Define the homomorphism

$$\Psi : Z^1(G, V) \to V^3; f \mapsto (f(\alpha), f(\tau), f(\nu)).$$

Then there are module isomorphisms

1. $Z^1(G, V) \cong \Psi(Z^1(G, V))$
2. $D^1(G, V) \cong \Psi(D^1(G, V))$.

Therefore we have

$$\mathrm{H}^1(G, V) \cong \Psi(Z^1(G, V))/\Psi(D^1(G, V)).$$

For $1 \leq i \leq 4$, let r_i denote the relations in the given presentation of G. The submodule $\Psi(Z^1(G, V))$ is the set of all tuples in V^3 which satisfy the relations $f(r_1) = f(r_2) = f(r_3) = f(r_4) = 0$ for $f \in Z^1(G, V)$. As for the submodule $\Psi(D^1(G, V))$ of $\Psi(Z^1(G, V))$, if f_v is a coboundary corresponding to $v \in V$, that is $f_v(g) = gv - v$, then $\Psi(f_v) = ((\alpha - 1)v, (\tau - 1)v, (\nu - 1)v)$. To compute $\mathrm{H}^1(\Gamma, V)$ for Γ a congruence subgroup of G one uses Shapiro's lemma

$$\mathrm{H}^1(G, \mathrm{Ind}_\Gamma^G(V)) \cong \mathrm{H}^1(\Gamma, V).$$

This is how $\mathrm{H}^1(\Gamma, V)$ is computed in [28] and in [32].

For $\mathbb{Q}(\sqrt{-1})$ data provided by Şengün suggest that computing with Manin symbols is more efficient than the direct computation of cohomology in degree 1. More precisely we have listed the dimensions of $Z^1(\Gamma_0(\mathfrak{n}), \mathbb{C}), D^1(\Gamma_0(\mathfrak{n}), \mathbb{C}), M_{\mathbb{C}}(\Gamma_0(\mathfrak{n}), \mathbb{C})$, the relation ideals J and the CPU time to compute $\mathrm{H}^1(\Gamma_0(\mathfrak{n}), \mathbb{C})$ and $\mathrm{H}^2(\Gamma_0(\mathfrak{n}), \mathbb{C})$. As already said from Table 4.5.1 below, we see that computing with modular symbols is efficient. This comparison is pertinent because the internal dimensions of $\mathrm{H}^1(\Gamma_0(\mathfrak{n}), \mathbb{C})$ and $\mathrm{H}^2(\Gamma_0(\mathfrak{n}), \mathbb{C})$ are almost the same.

One other information we would like to share concerns Table 4.5.2 and Table 4.5.3 below. For some rational primes p from 2 to 97, we have listed the CPU time for the computation of T_π on $\mathrm{H}^2(\Gamma_0(\mathfrak{n}), \mathbb{C})$ with $\pi | p$ in Table 4.5.2. These Hecke operators are described via Heilbronn-Merel matrices. As for Table 4.5.3, we have listed the CPU time for the computation of Hecke operators T_π on $\mathrm{H}^1(\Gamma_0(\mathfrak{n}), \mathbb{C})$, but these are described classically, i.e, not via Heilbronn-Merel matrices. We did not make precise the exact π because this does not alter the main information that one can draw from the experimentation. From Table 4.5.3, we see that classical computation of Hecke operators on $\mathrm{H}^1(\Gamma_0(\mathfrak{n}), \mathbb{C})$ can take some time especially when the norm of $\mathfrak{n}$ grows. In contrast in Table 4.5.2, it is particularly interesting to observe that the times to perform these Hecke operators on $\mathrm{H}^2(\Gamma_0(\mathfrak{n}), \mathbb{C})$ are very reasonable and this shows that describing Hecke operators on Manin symbols is not only of theoretical interest but also of computational interest.

Level $\mathfrak{n}$	Dimension $Z^1(\Gamma_0(\mathfrak{n}), \mathbb{C})$	Dimension of $D^1(\Gamma_0(\mathfrak{n}), \mathbb{C})$	CPU time in seconds for $\mathrm{H}^1(\Gamma_0(\mathfrak{n}), \mathbb{C})$	CPU time in seconds for $\mathrm{H}^2(\Gamma_0(\mathfrak{n}), \mathbb{C})$	Dimension of $M_{\mathbb{C}}(\Gamma_0(\mathfrak{n}), \mathbb{C})$	Dimension of J
$7 + 4w$	84	83	1.359	0.500	84	80
$8+ 2w$	108	107	1.313	0.250	108	102
$6 +6w$	120	119	1.563	0.375	120	112
$7+ 7w$	150	149	3.250	0.391	150	146
10	216	215	4.313	0.609	216	204
$9+ 5w$	162	161	4.406	0.547	162	158
$9+ 7w$	252	251	8.422	0.938	252	244
$11 + 3w$	253	251	8.359	0.953	252	243
$10+ 6w$	217	215	6.125	0.797	216	207
$11 + 4w$	138	137	9.297	1.891	138	136
12	242	239	7.125	1.109	240	226

Table 4.5.1: $\mathrm{H}^1(\Gamma_0(\mathfrak{n}), \mathbb{C})$ versus $\mathrm{H}^2(\Gamma_0(\mathfrak{n}), \mathbb{C})$ over $\mathbb{Q}(\sqrt{-1})$.

Hecke operator	Level $\mathfrak{n}$, Dimension of $\mathrm{H}^2(\Gamma_0(\mathfrak{n}), \mathbb{C})$									
	$4w$, 8	$3+4w$, 2	7, 2	$7+w$, 4	$2+5w$, 13	$5+5w$, 4	$8+3w$, 5	$8+4w$, 17	$9+3w$, 8	10, 8
2		0.080	0.060	0.040		0.070			0.170	
3	0.160	0.140	0.140				0.200			0.210
11	0.910	0.770	0.720	0.370	0.690	0.530	0.950	1.330		0.830
17	1.540	1.310	1.220		1.130	0.940	1.620	2.120	1.180	1.380
19	1.820	1.510	1.420	0.720	1.370	1.080	2.050	2.440	1.320	1.570
41	4.190		3.320	1.710	3.430	2.630		5.570	3.120	3.710
43	4.500	3.810	3.490	1.810	3.730	2.800	5.170	5.780	3.330	3.920
59	6.380	5.530	5.080	2.610	5.460	3.630	7.930	8.470	4.900	5.780
67	7.250	6.270	5.780	2.970	6.060	4.010	9.070	9.620	5.600	6.560
73	8.240	6.970	6.380	3.380	6.800	4.510	9.840	10.700	6.450	8.000
83	9.360	8.070	7.310	3.850	8.110	5.200	10.090	12.230	8.850	9.620
89	10.180	8.770	8.000	4.250	8.940	5.770	11.190	13.570	9.330	9.340
97	11.080	9.600	8.860	4.660	9.660	6.290	12.050	15.050	10.720	10.200

Table 4.5.2: CPU time in seconds of some Hecke operators on $\mathrm{H}^2(\Gamma_0(\mathfrak{n}), \mathbb{C})$ over $\mathbb{Q}(\sqrt{-2})$.

Hecke operator	Level $\mathfrak{n}$, Dimension of $\mathrm{H}^1(\Gamma_0(\mathfrak{n}), \mathbb{C})$									
	$4w$, 9	$3+4w$, 3	7, 3	$7+w$, 5	$2+5w$, 14	$5+5w$, 5	$8+3w$, 6	$8+4w$, 18	$9+3w$, 9	10, 9
2		0.220	0.270	0.290		0.500			1.870	
3	0.290	0.230	0.280				1.260			1.270
11	0.640	0.540	0.700	1.160	2.430	2.180	3.890	7.770		4.440
17	0.990	0.850	1.100		4.060	3.830	6.550	14.900	12.450	7.640
19	1.080	0.900	1.200	2.190	4.350	4.170	7.160	16.170	13.530	8.550
41	2.520		2.880	5.690	10.650	10.900		41.780	34.690	21.700
43	2.700	2.460	3.320	6.200	12.580	11.780	20.280	45.810	41.700	23.590
59	4.230	3.530	4.720	8.550	19.470	17.400	27.250	64.620	64.330	36.070
67	5.070	4.210	5.350	9.980	24.090	20.480	31.010	69.140	74.160	42.410
73	5.050	4.080	5.350	9.680	24.770	20.300	29.980	65.990	65.920	44.950
83	6.040	4.800	6.560	11.840	29.350	27.060	35.820	83.060	76.970	62.160
89	6.910	5.930	7.750	14.120	32.980	31.840	42.080	103.220 0	91.900	72.300
97	7.250	6.510	8.850	15.460	34.600	33.510	45.400	104.980 0	105.970	70.740

Table 4.5.3: CPU time in seconds of some Hecke operators on $\mathrm{H}^1(\Gamma_0(\mathfrak{n}), \mathbb{C})$ over $\mathbb{Q}(\sqrt{-2})$.

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